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Computing Dirichlet eigenvalues of the Schrödinger operator with a PT-symmetric optical potential

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Abstract

We provide estimates for the eigenvalues of non-self-adjoint Sturm–Liouville operators with Dirichlet boundary conditions for a shift of the special potential $4\cos^2 x + 4iV\sin 2x$ that is a PT-symmetric optical potential, especially when $|c| = |\sqrt{1-4V^2}| < 2$ or correspondingly $0 \le V < \sqrt{5}/2$. We obtain some useful equations for calculating Dirichlet eigenvalues also for $|c| \ge 2$ or equally $V \ge \sqrt{5}/2$. We discuss our results by comparing them with the periodic and antiperiodic eigenvalues of the Schrödinger operator. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we give a numerical example with error analysis.

Mathematics Subject Classification: 34L05; 34L15; 65L15

Keywords: Eigenvalue estimations; Dirichlet boundary conditions; PT-symmetric optical potentials

1 Introduction and preliminary facts

In this paper, we consider the operator $D(q)$ generated in $L_2[0,\pi]$ by the differential expression

$$-y''(x) + q(x)y(x) \tag{1}$$

and Dirichlet boundary conditions

$$y(\pi) = y(0) = 0, \tag{2}$$

where q is the PT-symmetric optical potential of the form

$$q(x) = (1+2V)e^{i2x} + (1-2V)e^{-i2x}, \quad V \ge 0, \tag{3}$$

which is a shift of $4\cos^2 x + 4iV\sin 2x$.

Some physically interesting results have been obtained by considering the optical potential (3). The detailed investigations of the periodic optical potentials were illustrated on (3) in the papers [9, 10]. For the first time, the mathematical explanations of the non-reality of the spectrum of the Hill operator $L(q)$, generated in $L_2(-\infty,\infty)$ by differential expression (1) with potential (3), for $V > 0.5$ and finding the threshold 0.5 (the first critical point V_1) were given by Makris et al. [9, 10]. Moreover, they sketched the real and imaginary parts of the first two bands using numerical methods for $V = 0.85$. Midya et al. [12] reduced the operator $L(q)$ to the Mathieu operator, and using the tabular values, they established that there is a second critical point $V_2 \sim 0.888437$ after which no parts of the first and second bands remain real.

Some of the most valuable results were given by Veliev [19, 20]. In [19], he gave a complete description, along with a mathematical proof, of the shape of the spectrum of the Hill operator $L(q)$ with potential (3), when V changes from $1/2$ to $\sqrt{5}/2$. Then, he extended his results for all $V > 1/2$ in [20].

The case $V = 1/2$ was considered for the first time by Gasymov [5], and it was proved that the spectrum of the Hill operator $L(q)$ is $[0,\infty)$. This case was also investigated in [6, 17].

Note that the optical potential (3) is a PT-symmetric potential. For the properties of the general PT-symmetric potentials, see [1, 13, 18, 21] and references therein. Here, we only note that the investigations of PT-symmetric periodic potentials were initiated by Bender et al. [2].

It was proved by Veliev [15, see Theorem 1 and (26)] that, if $ab = cd$, where a, b, c, and d are arbitrary complex numbers, then the Hill operators $L(q)$ and $L(p)$, generated in $L_2(-\infty,\infty)$ by expression (1) with the potentials $q(x) = ae^{-i2x} + be^{i2x}$ and $p(x) = ce^{-i2x} + de^{i2x}$, have the same Hill discriminant and hence the same Bloch eigenvalues and spectrum. Therefore, the investigations of the operators $L_t(q)$, for $t \in (-1,1]$, generated in $L_2[0,\pi]$ by the differential expression (1) and the boundary conditions

$$y(\pi) = e^{i\pi t}y(0), \qquad y'(\pi) = e^{i\pi t}y'(0), \tag{4}$$

can be reduced to the investigations of the operators $H_t(c)$, generated in $L_2[0,\pi]$ by differential expression (1) and the boundary conditions (4) with the potential

$$p(x) = ce^{2ix} + ce^{-2ix} = 2c\cos(2x), \tag{5}$$

where $c = \sqrt{1-4V^2}$. In particular, the eigenvalues of $L_0(q)$ and $L_1(q)$ are called the periodic and antiperiodic eigenvalues of the Hill operator $L(q)$, respectively. It was also proved by Veliev [16] that, if $c \neq 0$, then the number λ is an eigenvalue of multiplicity s of the operator $H_0(c)$, generated in $L_2[0,\pi]$ by expression (1) and the periodic boundary conditions with potential (5), if and only if it is an eigenvalue of multiplicity s either of the operator $D(c)$ or of the operator $N(c)$, where $D(c)$ and $N(c)$ are the operators generated in $L_2[0,\pi]$ by expression (1) and Dirichlet and Neumann boundary conditions, respectively, with potential (5). The statement continues to hold if $H_0(c)$ is replaced by $H_1(c)$, where $H_1(c)$ is the operator generated in $L_2[0,\pi]$ by expression (1) and the antiperiodic boundary conditions with potential (5). The eigenvalues of $H_0(c)$, $H_1(c)$, $D(c)$, and $N(c)$ are called periodic, antiperiodic, Dirichlet and Neumann eigenvalues of the Hill operator $H(c)$, generated in $L_2(-\infty,\infty)$ by expression (1) with potential (5), respectively.

Therefore, it is known that (see also Summary 3 of [19]), if $c \neq 0$, then any periodic eigenvalue is either a Dirichlet eigenvalue or a Neumann eigenvalue. Similarly, any antiperiodic eigenvalue is either a Dirichlet eigenvalue or a Neumann eigenvalue. For this reason, to consider the spectrum of the operator $D(q)$, we can use the properties of both the PT-symmetric potential (3) and the even potential (5). The eigenvalues of $D(q)$ or $D(c)$ are called Dirichlet eigenvalues, and they are denoted by $\lambda_n(q)$, for $n \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. We may also use the notation $\lambda_n(c)$, $n \in \mathbb{Z}^+$, for Dirichlet eigenvalues.

In this paper, we give estimates for Dirichlet eigenvalues and compare the results found with the periodic and antiperiodic eigenvalues, in particular, when $|c| < 2$ or correspondingly $0 \leq V < \sqrt{5}/2$. We also provide some useful equations for calculating Dirichlet eigenvalues for the case $|c| \geq 2$ or equally $V \geq \sqrt{5}/2$. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we give a numerical example for $c^2 = -2.157281295$ with error analysis using Rouche's theorem.

For ease of reading, we first present the main ideas of the proofs of the main results. To give estimates for the small Dirichlet eigenvalues, we prove (See Theorem 1) first that Dirichlet eigenvalues satisfy the equation

$$\lambda - n^2 + \frac{P_{2n}}{\pi} - \sum_{k=1}^{\infty} A_{k,n}(\lambda) = 0,$$

for $|c| < 2$ if n is odd and for $|c| < 3$ if n is even, and $n \geq 1$, where $P_k = \int_0^{\pi} p(x)\cos kx dx$, and the infinite series $A_{k,n}$ is defined in (11). We consider the antiperiodic Dirichlet (AD) eigenvalues λ_{2n-1}, for $n = 1, 2, \dots$, in Theorem 1, and the periodic Dirichlet (PD) eigenvalues λ_{2n}, for $n = 1, 2, \dots$, in Theorem 2. In particular, we consider the first Dirichlet eigenvalues λ_1 and λ_2 in Theorem 1(a) and Theorem 2 (a), respectively, and prove that for $|c| < 2$, λ_1 is the root of equation (12) lying in the disk $d_1 = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 2|c|\}$ and that for $|c| < 3$, λ_2 is the root of (15) lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda - 4| \leq 2|c|\}$. Then, to estimate eigenvalues numerically, we take finite summations instead of the infinite series in equations (12), (13), (15), and (16) and approximate the eigenvalues by the roots of the polynomials derived from the mth approximations (17)–(20), the way it was done by Veliev in [19].

Now, we state some preliminary facts. It is well known that the spectrum of the operator $D(q)$ is discrete, and for large enough n, there is one eigenvalue (counting with multiplicity) in the neighborhood of n^2. See the basic and detailed classical results in [3, 7, 8, 11] and references therein. The eigenvalues of the operators $D(0)$ are n^2, for $n \in \mathbb{Z}^+$, and all eigenvalues of $D(0)$ are simple.

It is also known that (see [4, 8]) if c is a real nonzero number, then all eigenvalues of the operator $H_t(c)$, generated in $L_2[0,\pi]$ by expression (1) and the boundary conditions (4) with potential (5), are real and simple. These results were stated more precisely in [19], as follows:

Summary 1 Let $0 < c < \infty$. Then, all the eigenvalues of $H_t(c)$, for all $t \in (-1, 1]$, are real and simple, and the spectrum of the Hill operator $H(c)$, generated in $L_2(-\infty, \infty)$ by expression (1) with potential (5), consists of the real intervals

$$\Gamma_1 := [\lambda_0(c), \mu_{-1}(c)], \qquad \Gamma_2 := [\mu_{+1}(c), \lambda_{-1}(c)],$$
$$\Gamma_3 := [\lambda_{+1}(c), \mu_{-2}(c)], \qquad \Gamma_4 := [\mu_{+2}(c), \lambda_{-2}(c)], \dots,$$

where $\lambda_0(c)$, $\lambda_{-n}(c)$, $\lambda_{+n}(c)$, for $n = 1, 2, \dots$ are the eigenvalues of $H_0(c)$, and $\mu_{-n}(c)$, $\mu_{+n}(c)$, for $n = 1, 2 \dots$ are the eigenvalues of $H_1(c)$, and the following inequalities hold:

$$\lambda_0(c) < \mu_{-1}(c) < \mu_{+1}(c) < \lambda_{-1}(c) < \lambda_{+1}(c) < \mu_{-2}(c) < \mu_{+2}(c) < \lambda_{-2}(c) < \lambda_{+2}(c) < \cdots .$$

The bands $\Gamma_1, \Gamma_2, \dots$ of the spectrum $\sigma(H(c))$ of $H(c)$ are separated by the gaps

$$\Delta_1 := \big(\mu_{-1}(c), \mu_{+1}(c)\big), \qquad \Delta_2 := \big(\lambda_{-1}(c), \lambda_{+1}(c)\big), \qquad \Delta_3 := \big(\mu_{-2}(c), \mu_{+2}(c)\big), \dots .$$

In other notation, $\Gamma_n = \{\gamma_n(t) : t \in [0, 1]\}$, where $\gamma_1(t), \gamma_2(t), \dots$ are the eigenvalues of $H_t(c)$, called as Bloch eigenvalues corresponding to the quasimomentum t and satisfying $\gamma_1(t) < \gamma_2(t) < \cdots$. The Bloch eigenvalue $\gamma_n(t)$ continuously depends on t, and $\gamma_n(-t) = \gamma_n(t)$. These statements continue to hold for $L_t(q)$ and $L(q)$ if $0 < V < 1/2$.

By Theorem 9 of [20], for complex values of c, the eigenvalues of the operator $H_0(c)$ lie in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|c|\}$, for $n = 0, 1, 2, \dots$ and $|c| < 3$. Moreover, the disk D_n, for $n \geq 2$, has no common points with another disk D_m, for $m \neq n$ and the boundary of the disk $D_{n,\epsilon} := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|c| + \epsilon\}$, for $n = 2, 3, \dots$, belongs to the resolvent set of the operator $H_0(c)$, for all $|c| < 3$, if ϵ is a sufficiently small positive number. It implies that the number of eigenvalues (counting the multiplicity) of $H_0(c)$ lying in $D_{n,\epsilon}$, for $n \geq 2$, are the same for all $|c| < 3$. Since $H_0(0)$ has two eigenvalues in $D_{n,\epsilon}$, for $n \geq 2$, the operator $H_0(c)$ has also two eigenvalues for $|c| < 3$. Letting ϵ tend to zero, we obtain that $H_0(c)$ has two eigenvalues (counting the multiplicity) in D_n, for $n \geq 2$ and $|c| < 3$. Similarly, we prove that $H_0(c)$ has 3 eigenvalues in $D_0 \cup D_1$. We denote them by λ_0, λ_{-1}, and λ_{+1}.

Similarly, $H_1(c)$ has two eigenvalues (counting the multiplicity) in $d_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| \leq 2|c|\}$, for $n = 1, 2, \dots$ and $|c| < 2$. We denote the $(2n)$th and $(2n+1)$th periodic eigenvalues by $\lambda_{-n}(c)$ and $\lambda_{+n}(c)$, for $n = 1, 2, \dots$; the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues by $\mu_{-n}(c)$ and $\mu_{+n}(c)$, for $n = 1, 2, \dots$, respectively.

Since a Dirichlet eigenvalue is either a periodic or an antiperiodic eigenvalue, we can use the relevant disks in these statements. In general, λ_{2n-1}, for $n = 1, 2, \dots$, is an antiperiodic eigenvalue, called an antiperiodic Dirichlet (AD) eigenvalue, and λ_{2n}, for $n = 1, 2, \dots$, is a periodic eigenvalue, called a periodic Dirichlet (PD) eigenvalue. In particular, since the first Dirichlet eigenvalue λ_1 is an antiperiodic eigenvalue and the second Dirichlet eigenvalue λ_2 is a periodic eigenvalue, λ_1 lies in the disk d_1 and λ_2 lies in the disk D_1. Thus,

$$\big|\lambda_n(c) - \lambda_n(0)\big| \leq 2|c|,$$

for $n \geq 1$, where $\lambda_n(0) = n^2$ and $c = \sqrt{1 - 4V^2}$. Therefore, we have

$$n^2 - 2|c| \leq |\lambda_n| \leq n^2 + 2|c|.$$

If $n = 2m$, for $m = 2, 3, \dots$, then

$$\begin{aligned}\big|\lambda_n - (2k)^2\big| &\geq \big|(2m)^2 - (2k)^2\big| - 2|c| = 4|m-k||m+k| - 2|c| \\ &\geq 4|2m-1| - 2|c|,\end{aligned}$$

for $|c| < 3$ and $k \neq \pm m$. Besides, if $m \geq 2$, we have $|\lambda_{2m}| \geq |\lambda_4| \geq 16 - 2|c| > 10$ and

$$\left|\lambda_n - (2k)^2\right| \geq \left||\lambda_4| - (2k)^2\right| \geq |\lambda_4| - 4 \geq 12 - 2|c| > 6, \tag{6}$$

for $|c| < 3$ and $k \neq \pm m$. In particular, if $m = 1$, we have $|\lambda_2| \leq 4 + 2|c| < 10$ and

$$\left|\lambda_2 - (2k)^2\right| \geq \left||\lambda_2| - (2k)^2\right| \geq 16 - |\lambda_2| \geq 12 - 2|c| > 6, \tag{7}$$

for $|c| < 3$ and $k \geq 2$. The analogous inequalities can be written for the case $n = 2m - 1$, from the inequalities

$$(2m-1)^2 - 2|c| \leq |\lambda_n| \leq (2m-1)^2 + 2|c|,$$

for $|c| < 2$ and $m = 1, 2, \ldots$. If $m = 1$, we have $|\lambda_1| \leq 1 + 2|c| < 5$ and

$$\left|\lambda_1 - (2k-1)^2\right| \geq \left||\lambda_1| - (2k-1)^2\right| \geq 9 - |\lambda_1| \geq 8 - 2|c| > 4, \tag{8}$$

for $|c| < 2$ and $k \geq 2$. Besides, if $m \geq 2$, we have $|\lambda_n| \geq |\lambda_3| \geq 9 - 2|c| > 5$ and

$$\left|\lambda_n - (2k-1)^2\right| \geq \left||\lambda_3| - (2k-1)^2\right| \geq |\lambda_3| - 1 \geq 8 - 2|c| > 4, \tag{9}$$

for $k \neq \pm m$.

2 Main results

We start with the equation

$$\left(\lambda_N - n^2\right)(\Psi_N, \sin nx) = (p\Psi_N, \sin nx), \tag{10}$$

which is obtained from

$$-\Psi_N''(x) + p(x)\Psi_N(x) = \lambda_N \Psi_N(x)$$

by multiplying both sides of the equality by $\sin nx$, where $\Psi_N(x)$ is the eigenfunction corresponding to the eigenvalue λ_N. Since the system of root functions $\{\sqrt{2}\sin kx/\sqrt{\pi} : k \in \mathbb{Z}^+\}$ of $D(0)$ forms an orthonormal basis for $L_2[0,\pi]$, we have the decomposition

$$\Psi_n = \sum_{k=1}^{\infty} \frac{2}{\pi}(\Psi_n, \sin kx)\sin kx.$$

Using the decomposition

$$\Psi_n(x) = \sum_{n_1 > -n}^{\infty} \frac{2}{\pi}\left(\Psi_n, \sin(n+n_1)x\right)\sin(n+n_1)x$$

of $\Psi_N(x)$ by the orthonormal basis $\{\sqrt{2}\sin(n+n_1)x/\sqrt{\pi} : n_1 > -n\}$ and iterating equation (10) m times for $N = n$, the way it was done in the paper [22], we obtain

$$\left(\lambda_n - n^2 + \frac{P_{2n}}{\pi} - \sum_{k=1}^{m} A_{k,n}(\lambda_n)\right)(\Psi_n, \sin nx) = R_m(\lambda_n), \tag{11}$$

where $P_k = \int_0^\pi p(x)\cos kxdx$,

$$A_{1,n}(\lambda) = \frac{1}{\pi^2}\sum_{n_1\neq 0,-2n}\frac{P_{n_1}(P_{n_1}-P_{n_1+2n})}{\lambda-(n+n_1)^2},$$

$$A_{k,n}(\lambda) = \frac{1}{\pi^{k+1}}\sum_{n_1,n_2,\dots,n_k}\frac{P_{n_1}P_{n_2}\cdots P_{n_k}(P_{n_1+n_2+\cdots+n_k}-P_{n_1+n_2+\cdots+n_k+2n})}{[\lambda-(n+n_1)^2]\cdots[\lambda-(n+n_1+\cdots+n_k)^2]},$$

$$R_m(\lambda) = \frac{1}{\pi^{m+1}}\sum_{n_1,n_2,\dots,n_{m+1}}\frac{P_{n_1}P_{n_2}\cdots P_{n_{m+1}}(P_{n_1+n_2+\cdots+n_{m+1}}-P_{n_1+n_2+\cdots+n_{m+1}+2n})}{[\lambda-(n+n_1)^2]\cdots[\lambda-(n+n_1+\cdots+n_{m+1})^2]}.$$

Here, the sums are taken under the conditions $n_s = \pm 2$, $\sum_{j=1}^{s} n_j \neq 0, -2n$ for $s = 1, 2, \dots, m+1$. Note that for the potential of the form (5), we have $P_2 = P_{-2} = c\pi$ and $P_k = 0$ for $k \neq \pm 2$.

We stress that the iteration formula (11) was used in [22] for large eigenvalues to obtain asymptotic formulas. In this paper, we find conditions on potentials (3) and (5) for which the iteration formula (11) is also valid for the small eigenvalues, as m tends to infinity. We also note that it is not easy to give such conditions, there are many technical calculations. Since the potential p is the even potential of the form (5), we have $A_{2k,2n}(\lambda_{2n}) = 0$, after some calculations, for $k = 1, 2, \dots$. Now, to give the main results, we prove the following lemmas. Without loss of generality, we assume that $\Psi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue λ_n.

First, we state the following lemma for AD eigenvalues λ_{2n-1}, for $n = 1, 2, \dots$:

Lemma 1 *The statements*

(a) $\lim_{m\to\infty} R_m(\lambda_{2n-1}) = 0$ *and* (b) $|(\Psi_{2n-1}, \sin(2n-1)x)|^2 > 0$

are valid in the following cases:

Case 1. *If* $|c| < 2$, *for all* $n \geq 1$,

Case 2. *If* $|c| < 2s$, *for* $n \geq 1+s$ *and* $s = 1, 2, \dots$.

Proof Case 1. (a) By the definition of $R_m(\lambda_n)$ and the conditions imposed on the summations, the number of summands of $R_{2m}(\lambda_n)$ is not greater than 4^m. On the other hand, since $\|\Psi_n\| = 1$ and $\|\sin kx\| = \sqrt{\pi}/\sqrt{2}$, by the Schwarz inequality, we have $|(p\Psi_n, \sin kx)| \leq \sqrt{2\pi}c$. First, we estimate $R_{2m}(\lambda_1)$, corresponding to the first Dirichlet eigenvalue λ_1. Considering the greatest summands of $R_{2m}(\lambda_1)$ in absolute value and taking (8)–(9) into account, we obtain

$$\begin{aligned}\left|R_{2m}(\lambda_1)\right| &< \frac{4^m|P_2|^{2m+1}|(p\Psi_1,\sin 3x)|}{\pi^{2m+1}|\lambda_1-9|^{m+1}|\lambda_1-25|^m} \leq 2|c|\frac{\sqrt{\pi}}{\sqrt{2}}\frac{4^m|c|^{2m+1}}{(8-2|c|)^{m+1}(24-2|c|)^m}\\ &< 4\frac{\sqrt{\pi}}{\sqrt{2}}\frac{4^m 2^{2m+1}}{4^{m+1}20^m} = \sqrt{2\pi}\left(\frac{1}{5}\right)^m,\end{aligned}$$

for $|c| < 2$. Similarly, for $n = 2$,

$$\begin{aligned}\left|R_{2m}(\lambda_3)\right| &< \frac{2^m|P_2|^{2m+1}|(p\Psi_3,\sin x)|}{\pi^{2m+1}|\lambda_3-1|^{2m+1}} + \frac{4^m|P_2|^{2m+1}|(p\Psi_3,\sin 5x)|}{\pi^{2m+1}|\lambda_3-25|^{m+1}|\lambda_3-49|^m}\\ &\leq 2|c|\frac{\sqrt{\pi}}{\sqrt{2}}\frac{2^m|c|^{2m+1}}{(8-2|c|)^{2m+1}} + 2|c|\frac{\sqrt{\pi}}{\sqrt{2}}\frac{4^m|c|^{2m+1}}{(16-2|c|)^{m+1}(40-2|c|)^m}\\ &< 4\frac{\sqrt{\pi}}{\sqrt{2}}\frac{2^m 2^{2m+1}}{4^{2m+1}} + 4\frac{\sqrt{\pi}}{\sqrt{2}}\frac{4^m 2^{2m+1}}{12^{m+1}36^m} = \sqrt{2\pi}\left(\frac{1}{2}\right)^m + \frac{\sqrt{2\pi}}{3}\left(\frac{1}{27}\right)^m,\end{aligned}$$

for $|c| < 2$. By the same way, for $n \geq 3$, we have

$$\begin{aligned}
\left|R_{2m}(\lambda_{2n-1})\right| &< \frac{4^m |P_2|^{2m+1} |(p\Psi_{2n-1}, \sin(2n-3)x)|}{\pi^{2m+1} |\lambda_{2n-1} - (2n-3)^2|^{m+1} |\lambda_{2n-1} - (2n-5)^2|^m} \\
&\leq 2|c| \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m |c|^{2m+1}}{(16 - 2|c|)^{m+1} (24 - 2|c|)^m} \\
&< 4 \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m 2^{2m+1}}{12^{m+1} 20^m} = \frac{\sqrt{2\pi}}{3} \left(\frac{1}{15}\right)^m,
\end{aligned}$$

for $|c| < 2$. Therefore, $\lim_{m\to\infty} R_m(\lambda_{2n-1}) = 0$, for all $n \geq 1$ and $|c| < 2$.

(b) Suppose the contrary, $(\Psi_{2n-1}, \sin(2n-1)x) = 0$. Since the system of root functions $\{\sqrt{2} \sin kx/\sqrt{\pi} : k \in \mathbb{Z}^+\}$ of $D(0)$ forms an orthonormal basis for $L_2[0, \pi]$, we have the decomposition

$$\frac{\pi}{2} \Psi_{2n-1} = \left(\Psi_{2n-1}, \sin(2n-1)x\right) \sin(2n-1)x + \sum_{k\in\mathbb{Z}^+, k\neq n} \left(\Psi_{2n-1}, \sin(2k-1)x\right) \sin(2k-1)x$$

for the normalized eigenfunction Ψ_{2n-1} corresponding to the eigenvalue λ_{2n-1} of $D(q)$. By Parseval's equality, we obtain

$$\sum_{k\in\mathbb{Z}^+, k\neq n} \left|\left(\Psi_{2n-1}, \sin(2k-1)x\right)\right|^2 = \frac{\pi}{2}.$$

First, we consider the case $n = 1$. Using relation (10) and the Bessel inequality and taking (8)–(9) into account, we obtain

$$\begin{aligned}
\frac{\pi}{2} &= \sum_{k\in\mathbb{Z}^+, k\neq 1} \left|\left(\Psi_1, \sin(2k-1)x\right)\right|^2 = \sum_{k\in\mathbb{Z}^+, k\neq 1} \frac{|(p\Psi_1, \sin(2k-1)x)|^2}{|\lambda_1 - (2k-1)^2|^2} \\
&\leq \frac{1}{(8 - 2|c|)^2} \sum_{k\in\mathbb{Z}, k\neq \pm 1} \left|\left(p\Psi_2, \sin(2k-1)x\right)\right|^2 < \frac{2|c|^2 \pi}{(8 - 2|c|)^2} < \frac{\pi}{2},
\end{aligned}$$

for $|c| < 2$, which is a contradiction. Similarly, in the case $n \geq 2$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^+, k\neq n} \left|\left(\Psi_{2n-1}, \sin(2k-1)x\right)\right|^2 &= \sum_{k\in\mathbb{Z}^+, k\neq n} \frac{|(p\Psi_{2n-1}, \sin(2k-1)x)|^2}{|\lambda_{2n-1} - (2k-1)^2|^2} \\
&\leq \frac{1}{(8 - 2|c|)^2} \sum_{k\in\mathbb{Z}, k\neq n} \left|\left(p\Psi_{2n-1}, \sin(2k-1)x\right)\right|^2 \\
&< \frac{2|c|^2 \pi}{(8 - 2|c|)^2} < \frac{\pi}{2},
\end{aligned}$$

for $|c| < 2$, which contradicts $\sum_{k\in\mathbb{Z}^+, k\neq n} |(\Psi_{2n-1}, \sin(2k-1)x)|^2 = \pi/2$ and completes the proof for Case 1.

Case 2. Now, consider the case $|c| < 2s$ and $n \geq 1 + s$ for $s \geq 1$. Using $(2n-1)^2 - 2|c| \leq |\lambda_{2n-1}| \leq (2n-1)^2 + 2|c|$, we obtain for $k \neq n$,

$$\left|\lambda_{2n-1} - (2k-1)^2\right| \geq \left|\lambda_{2n-1} - \left(2(n-1) - 1\right)^2\right| \geq (2n-1)^2 - 2|c| - (2n-3)^2$$

$$= 8n - 8 - 2|c| \geq 8(1+s) - 8 - 2(2s) = 4s,$$

and for $k \neq n, n-1$, we have

$$\left|\lambda_{2n-1} - (2k-1)^2\right| \geq \left|\lambda_{2n-1} - \big(2(n+1)-1\big)^2\right| \geq (2n+1)^2 - (2n-1)^2 - 2|c|$$
$$= 8n - 2|c| \geq 8(1+s) - 2(2s) = 4s + 8.$$

Therefore, for $|c| < 2s$, $n \geq 1+s$ and $s = 1, 2, \ldots$, using these inequalities and arguing as in the proof of (a) for Case 1, we complete the proof of (a) for Case 2.

For the proof of (b), again assume the contrary $(\Psi_{2n-1}, \sin(2n-1)x) = 0$. Then,

$$\frac{\pi}{2} = \sum_{k\in\mathbb{Z}^+, k\neq n} \left|\big(\Psi_{2n-1}, \sin(2k-1)x\big)\right|^2 = \sum_{k\in\mathbb{Z}^+, k\neq n+1} \frac{|(p\Psi_{2n-1}, \sin(2k-1)x)|^2}{|\lambda_{2n-1} - (2k-1)^2|^2}$$
$$\leq \frac{1}{(4s)^2} \sum_{k\in\mathbb{Z}, k\neq n+1} \left|\big(p\Psi_{2n+1}, \sin(2k-1)x\big)\right|^2 < \frac{2|c|^2\pi}{(4s)^2} < \frac{8s^2\pi}{16s^2} < \frac{\pi}{2},$$

and for $|c| < 2s-1$, $n \geq s$ and $s = 2, 3, \ldots$, which contradicts $\sum_{k\in\mathbb{Z}^+, k\neq n} |(\Psi_{2n-1}, \sin(2k-1)x)|^2 = \pi/2$ and completes the proof for Case 2. □

Now we state the analogous lemma for PD eigenvalues λ_{2n}, for $n = 1, 2, \ldots$:

Lemma 2 *The statements*

(a) $\lim_{m\to\infty} R_m(\lambda_{2n}) = 0$ *and* (b) $|(\Psi_{2n}, \sin 2nx)|^2 > 0$

are valid in the following cases:

Case 1. *If* $|c| < 3$, *for all* $n \geq 1$,

Case 2. *If* $|c| < 2s-1$, *for* $n \geq s$ *and* $s = 2, 3, \ldots$.

Proof Case 1. (a) Arguing as in the proof of Lemma 1, first, we estimate $R_{2m}(\lambda_2)$, corresponding to the second Dirichlet eigenvalue λ_2. Considering the greatest summands of $R_{2m}(\lambda_2)$ in absolute value and taking (6)–(7) into account, we obtain

$$\left|R_{2m}(\lambda_2)\right| < \frac{4^m |P_2|^{2m+1} |(p\Psi_2, \sin 4x)|}{\pi^{2m+1} |\lambda_2 - 16|^{m+1} |\lambda_2 - 36|^m} \leq 2|c| \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m |c|^{2m+1}}{(12 - 2|c|)^{m+1} (32 - 2|c|)^m}$$
$$< 6 \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m 3^{2m+1}}{6^{m+1} 26^m} < \frac{3\sqrt{2\pi}}{2} \left(\frac{3}{13}\right)^m,$$

for $|c| < 3$. Similarly, for $n \geq 2$, we have

$$\left|R_{2m}(\lambda_{2n})\right| < \frac{4^m |P_2|^{2m+1} |(p\Psi_{2n}, \sin(2n-2)x)|}{\pi^{2m+1} |\lambda_{2n} - (2n-2)^2|^{m+1} |\lambda_{2n} - (2n-4)^2|^m}$$
$$\leq 2|c| \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m |c|^{2m+1}}{(12 - 2|c|)^{m+1} (16 - 2|c|)^m} < 6 \frac{\sqrt{\pi}}{\sqrt{2}} \frac{4^m 3^{2m+1}}{6^{m+1} 10^m} < \frac{3\sqrt{2\pi}}{2} \left(\frac{3}{5}\right)^m,$$

for $|c| < 3$. Therefore, $\lim_{m\to\infty} R_m(\lambda_{2n}) = 0$, for all $n \geq 1$ and $|c| < 3$.

(b) Suppose the contrary, $(\Psi_{2n}, \sin 2nx) = 0$. Since the system of root functions $\{\sqrt{2}\sin kx/\sqrt{\pi} : k \in \mathbb{Z}^+\}$ of $D(0)$ forms an orthonormal basis for $L_2[0, \pi]$, we have the

decomposition

$$\frac{\pi}{2}\Psi_{2n} = (\Psi_{2n}, \sin 2nx)\sin 2nx + \sum_{k\in\mathbb{Z}^+, k\neq n} (\Psi_{2n}, \sin 2kx)\sin 2kx$$

for the normalized eigenfunction Ψ_{2n} corresponding to the eigenvalue λ_{2n} of $D(q)$. By Parseval's equality, we obtain

$$\sum_{k\in\mathbb{Z}^+, k\neq n} \left|(\Psi_{2n}, \sin 2kx)\right|^2 = \frac{\pi}{2}.$$

First, we consider the case $n = 1$. Using relation (10) and the Bessel inequality and taking into account (6)–(7), we obtain

$$\begin{aligned}\frac{\pi}{2} &= \sum_{k\in\mathbb{Z}^+, k\neq 2} \left|(\Psi_2, \sin 2kx)\right|^2 = \sum_{k\in\mathbb{Z}^+, k\neq 1} \frac{|(p\Psi_2, \sin 2kx)|^2}{|\lambda_2 - (2k)^2|^2} \\ &\leq \frac{1}{(12-2|c|)^2} \sum_{k\in\mathbb{Z}, k\neq 1} \left|(p\Psi_2, \sin 2kx)\right|^2 < \frac{2|c|^2\pi}{(12-2|c|)^2} < \frac{\pi}{2},\end{aligned}$$

for $|c| < 3$, which is a contradiction. Similarly, in the case $n \geq 2$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^+, k\neq n} \left|(\Psi_{2n}, \sin 2kx)\right|^2 &= \sum_{k\in\mathbb{Z}^+, k\neq n} \frac{|(p\Psi_{2n}, \sin 2kx)|^2}{|\lambda_{2n} - (2k)^2|^2} \\ &\leq \frac{1}{(12-2|c|)^2} \sum_{k\in\mathbb{Z}, k\neq \pm n} \left|(p\Psi_{2n}, \sin 2kx)\right|^2 < \frac{2|c|^2\pi}{(12-2|c|)^2} < \frac{\pi}{2},\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z}^+, k\neq n} |(\Psi_{2n}, \sin 2kx)|^2 = \pi/2$ and completes the proof for Case 1.

Case 2. Now, consider the case $|c| < 2s - 1$ and $n \geq s$ for $s \geq 2$. Using $(2n)^2 - 2|c| \leq |\lambda_n| \leq (2n)^2 + 2|c|$, we obtain for $k \neq n$,

$$\begin{aligned}\left|\lambda_{2n} - (2k)^2\right| &\geq \left|\lambda_{2n} - \bigl(2(n-1)\bigr)^2\right| \geq (2n)^2 - 2|c| - \bigl(2(n-1)\bigr)^2 \\ &= 4(2n-1) - 2|c| \geq 4(2s-1) - 2(2s-1) = 4s - 2,\end{aligned}$$

and for $k \neq n, n-1$, we have

$$\begin{aligned}\left|\lambda_{2n} - (2k)^2\right| &\geq \left|\lambda_{2n} - \bigl(2(n+1)\bigr)^2\right| \geq \bigl(2(n+1)\bigr)^2 - (2n)^2 - 2|c| \\ &= 4(2n+1) - 2|c| \geq 4(2s+1) - 2(2s-1) = 4s + 6.\end{aligned}$$

Therefore, for the case $|c| < 2s - 1$, $n \geq s$ and $s = 2, 3, \ldots$, using these inequalities and arguing as in the proof of (a) for Case 1, we complete the proof of (a) for Case 2.

For the proof of (b), again suppose the contrary $(\Psi_{2n-1}, \sin(2n-1)x) = 0$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^+, k\neq n} \left|(\Psi_{2n}, \sin 2kx)\right|^2 &= \sum_{k\in\mathbb{Z}^+, k\neq n} \frac{|(p\Psi_{2n}, \sin 2kx)|^2}{|\lambda_{2n} - (2k)^2|^2} \\ &\leq \frac{1}{(4s-2)^2} \sum_{k\in\mathbb{Z}, k\neq \pm n} \left|(p\Psi_{2n}, \sin 2kx)\right|^2 < \frac{2|c|^2\pi}{(4s-2)^2}\end{aligned}$$

$$< \frac{2(2s-1)^2\pi}{(4s-2)^2} < \frac{\pi}{2},$$

for $|c| < 2s-1$, $n \geq s$ and $s = 2,3,\ldots$, which contradicts $\sum_{k\in\mathbb{Z}^+,k\neq n} |(\Psi_{2n}, \sin 2kx)|^2 = \pi/2$ and completes the proof for Case 2. □

Now, letting m tend to infinity in equation (11), we obtain the following results. First, we consider the antiperiodic Dirichlet (AD) eigenvalues λ_{2n-1}, for $n = 1,2,\ldots$:

Theorem 1 (a) *If $|c| < 2$, then the first antiperiodic Dirichlet eigenvalue λ_1 is the root of*

$$\lambda - 1 + c - \frac{c^2}{\lambda - 9} - \sum_{k=1}^{\infty} A_{2k+1,1}(\lambda) = 0, \tag{12}$$

lying in the disk $d_1 = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 2|c|\}$, where $A_{k,n}$ is defined in (11), and the series $\sum_{k=1}^{\infty} A_{2k+1,1}(\lambda)$ converges uniformly to an analytic function on the disk d_1. Moreover, (12) has exactly one root (counting with multiplicity) in d_1, and this root coincides with the first Dirichlet eigenvalue λ_1.

(b) *If $|c| < 2$ and $n \geq 2$, then λ_{2n-1} is an eigenvalue of $D(q)$ if and only if it is the root of the equation*

$$\lambda - (2n-1)^2 - \sum_{k=1}^{\infty} A_{k,2n-1}(\lambda) = 0, \tag{13}$$

lying in the disk $d_n := \{\lambda \in \mathbb{C} : |\lambda - (2n-1)^2| \leq 2|c|\}$ and the series $\sum_{k=1}^{\infty} A_{k,2n-1}(\lambda)$ converges uniformly to an analytic function on the disk d_n.

(c) *If $|c| < 2s$, then the statements of (b) are still valid for $n \geq 1 + s$ and $s = 1,2,\ldots$.*

Proof (a) By Lemma 1, letting m tend to infinity in the equation (11), we obtain

$$\lambda_n - n^2 + \frac{P_{2n}}{\pi} - \sum_{k=1}^{\infty} A_{k,n}(\lambda_n) = 0, \tag{14}$$

where $P_k = \int_0^{\pi} p(x)\cos kx dx$, $P_2 = c\pi$, $P_k = 0$ for $k \neq \pm 2$. For $n = 1$, we have $A_{2k,1}(\lambda_1) = 0$, for $k = 1,2,\ldots$, and $A_{1,1}(\lambda_1) = c^2/(\lambda - 9)$. Substituting these values in (14), we obtain (12).

Now, we prove that the root of (12) lying in the disk d_1 is an eigenvalue of the operator D. The equation $f_1(\lambda) := \lambda - 1 + c - c^2/(\lambda - 9) = 0$ has one root in the disk d_1 and

$$\begin{aligned}
|f_1(\lambda)| &= \left|\lambda - 1 + c - \frac{c^2}{\lambda - 9}\right| \geq \left||\lambda - 1| - |c| - \frac{|c|^2}{|\lambda - 9|}\right| \\
&\geq 2|c| - |c| - \frac{|c|^2}{8 - 2|c|} = |c| - \frac{|c|^2}{8 - 2|c|},
\end{aligned}$$

for all $\lambda \in c_1 := \{\lambda \in \mathbb{C} : |\lambda - 1| = 2|c|\}$. We define

$$g_1(\lambda) := \lambda - 1 + c - \frac{c^2}{\lambda - 9} - \sum_{k=1}^{\infty} A_{2k+1,1}(\lambda).$$

Estimating the summands of $|A_{2k+1,1}(\lambda_1)|$, we obtain

$$\left|A_{2k+1,1}(\lambda_1)\right| < \frac{2^{k-1}|c|^{2k+2}}{|\lambda_1-9|^{k+1}|\lambda_1-25|^k} \le \frac{2^{k-1}|c|^{2k+2}}{(8-2|c|)^{k+1}(24-2|c|)^k},$$

for $k \ge 1$, and by the geometric series formula, we obtain

$$\sum_{k=1}^{\infty}\left|A_{2k+1,1}(\lambda_1)\right| < \frac{|c|^4}{(8-2|c|)[(8-2|c|)(24-2|c|)-2|c|^2]} < \frac{1}{18}.$$

Hence

$$\begin{aligned}\left|g_1(\lambda)-f_1(\lambda)\right| &= \left|\sum_{k=1}^{\infty}A_{2k+1,1}(\lambda)\right| \le \sum_{k=1}^{\infty}\left|A_{2k+1,1}(\lambda)\right| \\ &< \frac{|c|^4}{(8-2|c|)[(8-2|c|)(24-2|c|)-2|c|^2]} < \frac{1}{18},\end{aligned}$$

for all $\lambda \in c_1$. Therefore, $|g_1(\lambda)-f_1(\lambda)| < |f_1(\lambda)|$ holds for all $\lambda \in c_1$, and by Rouche's theorem, $g_1(\lambda)$ has one root in the disk d_1. Hence, the operator D has one eigenvalue (counting with multiplicity) lying in d_1, which is the root of (12). On the other hand, equation (12) has exactly one root (counting with multiplicity) in d_1. Thus, $\lambda \in d_1$ is an eigenvalue of D if and only if it is the root of (12) and the root of (12) coincides with the eigenvalue λ_1 of D.

Now, to estimate $\sum_{k=1}^{\infty}|A'_{2k+1,1}(\lambda)|$, for $|\lambda-1| \le 2|c|$ and $|c| < 2$, we first estimate the summands of $|A'_{2k+1,1}(\lambda_1)|$ by differentiating $A_{2k+1,1}(\lambda_1)$ with respect to λ_1:

$$\left|A'_{2k+1,1}(\lambda_1)\right| < \frac{3^k|c|^{2k+2}}{|\lambda_1-9|^{k+2}|\lambda_1-25|^k} \le \frac{3^k|c|^{2k+2}}{(8-2|c|)^{k+2}(24-2|c|)^k},$$

for $k \ge 1$, and by the geometric series formula, we obtain

$$\sum_{k=1}^{\infty}\left|A'_{2k+1,1}(\lambda_1)\right| < \frac{3|c|^4}{(8-2|c|)^2[(8-2|c|)(24-2|c|)-3|c|^2]} < \frac{3}{51}.$$

Therefore, the series $\sum_{k=1}^{\infty}A_{2k+1,1}(\lambda)$ converges uniformly to an analytic function on the disk d_1.

(b) Let $f_n(\lambda) := \lambda-(2n-1)^2-A_{1,2n-1}(\lambda)-A_{2,2n-1}(\lambda)$ and

$$g_n(\lambda) := \lambda-(2n-1)^2-A_{1,2n-1}(\lambda)-A_{2,2n-1}(\lambda)-\sum_{k=3}^{\infty}A_{k,2n-1}(\lambda).$$

Then, $f_n(\lambda)$ has one root in the disk d_n and

$$\begin{aligned}\left|f_n(\lambda)\right| &= \left|\lambda-(2n-1)^2-A_{1,2n-1}(\lambda)-A_{2,2n-1}(\lambda)\right| \\ &\ge \left|\left|\lambda-(2n-1)^2\right|-\left|A_{1,2n-1}(\lambda)\right|-\left|A_{2,2n-1}(\lambda)\right|\right| \\ &\ge \left|\left|\lambda-(2n-1)^2\right|-\left|A_{1,3}(\lambda_3)\right|-\left|A_{2,3}(\lambda_3)\right|\right| \\ &\ge 2|c|-\frac{|c|^2}{8-2|c|}-\frac{|c|^2}{16-2|c|}-\frac{|c|^3}{(8-2|c|)^2},\end{aligned}$$

for all $\lambda \in c_n := \{\lambda \in \mathbb{C} : |\lambda - (2n-1)^2| = 2|c|\}$, where

$$A_{1,3}(\lambda_3) = \frac{c^2}{\lambda_3 - 1} + \frac{c^2}{\lambda_3 - 25}, \qquad A_{2,3}(\lambda_3) = -\frac{c^3}{(\lambda_3 - 1)^2}.$$

Using the estimates for $|A_{k,3}(\lambda_3)|$,

$$\begin{aligned}
\left|A_{2k+1,3}(\lambda_3)\right| &< \frac{|c|^{2k+2}}{|\lambda_3 - 1|^{2k+1}} + \frac{(3/2)^{k-1}|c|^{2k+2}}{|\lambda_3 - 25|^{k+1}|\lambda_3 - 49|^k} \\
&\leq \frac{|c|^{2k+2}}{(8-2|c|)^{2k+1}} + \frac{(3/2)^{k-1}|c|^{2k+2}}{(16-2|c|)^{k+1}(40-2|c|)^k}, \\
\left|A_{2k,3}(\lambda_3)\right| &< \frac{|c|^{2k+1}}{|\lambda_3 - 1|^{2k}} \leq \frac{|c|^{2k+1}}{(8-2|c|)^{2k}},
\end{aligned}$$

for $\sum_{k=3}^{\infty} |A_{k,3}(\lambda_3)|$,

$$\begin{aligned}
\sum_{k=3}^{\infty}\left|A_{k,3}(\lambda_3)\right| &< \frac{|c|^4}{(8-2|c|)^2(8-3|c|)} + \frac{2|c|^4}{(16-2|c|)[2(16-2|c|)(40-2|c|)-3|c|^2]} \\
&< \frac{1}{2} + \frac{2}{639} = \frac{643}{1278},
\end{aligned}$$

for the derivative of $A_{k,3}(\lambda_3)$ with respect to λ_3,

$$\begin{aligned}
\left|A'_{2k+1,3}(\lambda_3)\right| &< \frac{(3/2)^{2k+1}|c|^{2k+2}}{|\lambda_3 - 1|^{2k+2}} + \frac{(3/2)^{2k+1}|c|^{2k+2}}{|\lambda_3 - 25|^{k+2}|\lambda_3 - 49|^k} \\
&\leq \frac{(3/2)^{2k+1}|c|^{2k+2}}{(8-2|c|)^{2k+2}} + \frac{(3/2)^{2k+1}|c|^{2k+2}}{(16-2|c|)^{k+2}(40-2|c|)^k}, \\
\left|A'_{2k,3}(\lambda_3)\right| &< \frac{(3/2)^{2k}|c|^{2k+1}}{|\lambda_3 - 1|^{2k+1}} \leq \frac{(3/2)^{2k}|c|^{2k+1}}{(8-2|c|)^{2k+1}},
\end{aligned}$$

and finally for $\sum_{k=3}^{\infty} |A'_{k,3}(\lambda_3)|$,

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|A'_{k+2,3}(\lambda_3)\right| &< \frac{27|c|^4}{4(8-2|c|)^3(16-7|c|)} \\
&\quad + \frac{27|c|^4}{2(16-2|c|)^2[4(16-2|c|)(40-2|c|)-9|c|^2]} < \frac{27}{32} + \frac{1}{8.47} < \frac{17}{20},
\end{aligned}$$

and arguing as in the proof of (a), we obtain the proof of (b).

(c) It is obvious by Lemma 1 (Case 2). □

Now, we consider the periodic Dirichlet (PD) eigenvalues λ_{2n}, for $n = 1, 2, \ldots$:

Theorem 2 (a) *If* $|c| < 3$, *then the first periodic Dirichlet eigenvalue* λ_2 *is the root of*

$$\lambda - 4 - \frac{c^2}{\lambda - 16} - \sum_{k=1}^{\infty} A_{2k+1,2}(\lambda) = 0, \tag{15}$$

lying in the disk $D_1 := \{\lambda \in \mathbb{C} : |\lambda - 4| \le 2|c|\}$, *and the series* $\sum_{k=1}^{\infty} A_{2k+1,2}(\lambda)$ *converges uniformly to an analytic function on the disk* D_1. *Moreover*, (15) *has exactly one root* (*counting with multiplicity*) *in* D_1, *and this root coincides with the eigenvalue* λ_2 *of the operator* D.

(b) *If* $|c| < 3$ *and* $n \ge 2$, *then* λ_{2n} *is an eigenvalue of* $D(q)$ *if and only if it is the root of the equation*

$$\lambda - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1,2n}(\lambda) = 0, \tag{16}$$

lying in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \le 2|c|\}$, *and the series* $\sum_{k=1}^{\infty} A_{2k-1,2n}(\lambda)$ *converges uniformly to an analytic function on the disk* D_n.

(c) *If* $|c| < 2s - 1$, *then the statements of* (*b*) *are still valid for* $n \ge s$ *and* $s = 2, 3, \ldots$.

Proof (a) Let $F_1(\lambda) := \lambda - 4 - \frac{c^2}{\lambda - 16}$ and

$$G_1(\lambda) := \lambda - 4 - \frac{c^2}{\lambda - 16} - \sum_{k=1}^{\infty} A_{2k+1,2}(\lambda).$$

Then, $F_1(\lambda)$ has one root in the disk D_1, and

$$\begin{aligned}\left|F_1(\lambda)\right| &= \left|\lambda - 4 - \frac{c^2}{\lambda - 16}\right| \ge \left||\lambda - 4| - \frac{|c|^2}{|\lambda - 16|}\right| \\ &\ge 2|c| - \frac{|c|^2}{12 - 2|c|},\end{aligned}$$

for all $\lambda \in C_1 := \{\lambda \in \mathbb{C} : |\lambda - 4| = 2|c|\}$. Using the estimates for $|A_{2k+1,2}(\lambda_2)|$,

$$\left|A_{2k+1,2}(\lambda_2)\right| < \frac{2^{k-1}|c|^{2k+2}}{|\lambda_2 - 16|^{k+1}|\lambda_2 - 36|^k} \le \frac{2^{k-1}|c|^{2k+2}}{(12 - 2|c|)^{k+1}(32 - 2|c|)^k},$$

for $\sum_{k=1}^{\infty} |A_{2k+1,2}(\lambda_2)|$,

$$\sum_{k=1}^{\infty}\left|A_{2k+1,2}(\lambda_2)\right| < \frac{|c|^4}{(12 - 2|c|)[(12 - 2|c|)(32 - 2|c|) - 2|c|^2]} < \frac{9}{92},$$

for the derivative of $A_{2k+1,2}(\lambda_2)$ with respect to λ_2,

$$\left|A'_{2k+1,2}(\lambda_2)\right| < \frac{3^k|c|^{2k+2}}{|\lambda_2 - 16|^{k+2}|\lambda_2 - 36|^k} \le \frac{3^k|c|^{2k+2}}{(12 - 2|c|)^{k+2}(32 - 2|c|)^k},$$

and finally for $\sum_{k=1}^{\infty} |A'_{2k+1,2}(\lambda_2)|$,

$$\sum_{k=1}^{\infty}\left|A'_{2k+1,2}(\lambda_2)\right| < \frac{3|c|^4}{(12 - 2|c|)^2[(12 - 2|c|)(32 - 2|c|) - 3|c|^2]} < \frac{9}{172},$$

and arguing as in the proof of Lemma 1(a), we obtain the proof of (a).

(b) Let $F_n(\lambda) := \lambda - (2n)^2 - A_{1,2n}(\lambda)$ and

$$G_n(\lambda) := \lambda - (2n)^2 - A_{1,2n}(\lambda) - \sum_{k=1}^{\infty} A_{2k+1,2n}(\lambda).$$

Then, $F_n(\lambda)$ has one root in the disk D_n and

$$\begin{aligned} \left|F_n(\lambda)\right| &= \left|\lambda - (2n)^2 - A_{1,2n}(\lambda)\right| \geq \left|\left|\lambda - (2n)^2\right| - \left|A_{1,2n}(\lambda)\right|\right| \\ &\geq \left|\left|\lambda - (2n)^2\right| - \left|A_{1,4}(\lambda_4)\right|\right| \geq 2|c| - \frac{|c|^2}{12 - 2|c|} - \frac{|c|^2}{20 - 2|c|}, \end{aligned}$$

for all $\lambda \in C_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| = 2|c|\}$, where

$$A_{1,4}(\lambda_4) = \frac{c^2}{\lambda_4 - 4} + \frac{c^2}{\lambda_4 - 36}.$$

Using the estimates for $|A_{2k+1,4}(\lambda_4)|$,

$$\left|A_{2k+1,4}(\lambda_4)\right| < \frac{(3/2)^{k-1}|c|^{2k+2}}{|\lambda_4 - 36|^{k+1}|\lambda_4 - 64|^k} \leq \frac{(3/2)^{k-1}|c|^{2k+2}}{(20 - 2|c|)^{k+1}(48 - 2|c|)^k},$$

for $\sum_{k=1}^{\infty} |A_{2k+1,4}(\lambda_4)|$,

$$\sum_{k=1}^{\infty}\left|A_{2k+1,4}(\lambda_4)\right| < \frac{2|c|^4}{(20 - 2|c|)[2(20 - 2|c|)(48 - 2|c|) - 3|c|^2]} < \frac{1}{99},$$

for the derivative of $A_{2k+1,4}(\lambda_4)$ with respect to λ_4,

$$\left|A'_{2k+1,4}(\lambda_4)\right| < \frac{(3/2)^{k+2}|c|^{2k+2}}{|\lambda_4 - 36|^{k+2}|\lambda_4 - 64|^k} \leq \frac{(3/2)^{k+2}|c|^{2k+2}}{(20 - 2|c|)^{k+2}(48 - 2|c|)^k},$$

and finally for $\sum_{k=1}^{\infty} |A'_{2k+1,4}(\lambda_4)|$,

$$\sum_{k=1}^{\infty}\left|A'_{2k+1,4}(\lambda_4)\right| < \frac{27|c|^4}{4(20 - 2|c|)^2[2(20 - 2|c|)(48 - 2|c|) - 3|c|^2]} < \frac{1}{400},$$

and arguing as in the proof of Lemma 1(a), we obtain the proof of (b).

(b) It is obvious by Lemma 2 (Case 2). □

Now, to estimate eigenvalues numerically, we take finite summations instead of the infinite series in the equations (12), (13), (15), and (16). If we consider the mth approximation

$$\lambda - 1 + c - \frac{c^2}{\lambda - 9} - \sum_{k=1}^{m} A_{2k+1,1}(\lambda) = 0, \tag{17}$$

for the first eigenvalue λ_1, the mth approximation

$$\lambda - 4 - \frac{c^2}{\lambda - 16} - \sum_{k=1}^{m} A_{2k+1,2}(\lambda) = 0 \tag{18}$$

for the second eigenvalue λ_2, and the mth approximations

$$\lambda - (2n-1)^2 - \sum_{k=1}^{2m} A_{k,2n-1}(\lambda) = 0, \tag{19}$$

and

$$\lambda - (2n)^2 - \sum_{k=1}^{m} A_{2k-1,2n}(\lambda) = 0, \tag{20}$$

for the other eigenvalues λ_{2n-1} and λ_{2n}, $n = 2, 3, \ldots$, of D, then we have the following estimates for the remaining terms:

$$\begin{aligned}\left|\sum_{k=m+1}^{\infty} A_{2k+1,1}(\lambda_1)\right| &\le \sum_{k=m+1}^{\infty} \left|A_{2k+1,1}(\lambda_1)\right| \\ &< \frac{2^m|c|^{2m+4}}{(8-2|c|)^{m+1}(24-2|c|)^m[(8-2|c|)(24-2|c|)-2|c|^2]} \\ &< \frac{1}{18}\left(\frac{1}{10}\right)^m,\end{aligned}$$

for $|c| < 2$,

$$\begin{aligned}\left|\sum_{k=m+1}^{\infty} A_{2k+1,2}(\lambda_2)\right| &\le \sum_{k=m+1}^{\infty} \left|A_{2k+1,2}(\lambda_2)\right| \\ &< \frac{2^m|c|^{2m+4}}{(12-2|c|)^{m+1}(32-2|c|)^m[(12-2|c|)(32-2|c|)-2|c|^2]} \\ &< \frac{9}{92}\left(\frac{3}{26}\right)^m,\end{aligned}$$

for $|c| < 3$,

$$\begin{aligned}\left|\sum_{k=2m+1}^{\infty} A_{k,2n-1}(\lambda_{2n-1})\right| &\le \sum_{k=2m+1}^{\infty} \left|A_{k,3}(\lambda_3)\right| \\ &< \frac{|c|^{2m+2}}{(8-2|c|)^{2m}(8-3|c|)} \\ &\quad + \frac{2(3/2)^{m-1}|c|^{2m+2}}{(16-2|c|)^m(40-2|c|)^{m-1}[2(16-2|c|)(40-2|c|)-3|c|^2]} \\ &< \frac{2}{4^m} + \frac{16}{71}\left(\frac{1}{72}\right)^m,\end{aligned}$$

for $|c| < 2$, and

$$\begin{aligned}&\left|\sum_{k=m+1}^{\infty} A_{2k-1,2n}(\lambda_{2n})\right| \\ &\quad \le \sum_{k=m+1}^{\infty} \left|A_{2k-1,4}(\lambda_4)\right|\end{aligned}$$

$$< \frac{2(3/2)^m|c|^{2m+4}}{(20-2|c|)^{m+1}(48-2|c|)^m[2(20-2|c|)(48-2|c|)-3|c|^2]} < \frac{27}{2681}\left(\frac{9}{392}\right)^m,$$

for $|c| < 3$. Obviously, we obtain better approximations as m grows. Besides, for a fixed m, this method gives better approximations as n grows. Now, we approach the eigenvalues by the roots of the polynomials derived from the mth approximations (17)–(20), as it was done in [19]. For example, for $m = 2$ and $n = 1$ in (17), we have the approximation

$$\begin{aligned} Q_1(\lambda) := \lambda - 1 + c - \frac{c^2}{\lambda - 9} - \frac{c^4}{(\lambda-9)^2(\lambda-25)} \\ - \frac{c^6}{(\lambda-9)^2(\lambda-25)^2(\lambda-49)} - \frac{c^6}{(\lambda-9)^3(\lambda-25)^2} = 0, \end{aligned} \tag{21}$$

for $m = 2$ and $n = 2$ in (18),

$$\begin{aligned} Q_2(\lambda) := \lambda - 4 - \frac{c^2}{\lambda - 16} - \frac{c^4}{(\lambda-16)^2(\lambda-36)} \\ - \frac{c^6}{(\lambda-16)^2(\lambda-36)^2(\lambda-64)} - \frac{c^6}{(\lambda-16)^3(\lambda-36)^2} = 0, \end{aligned} \tag{22}$$

for $m = 3$ and $n = 3$ in (19), we have

$$\begin{aligned} Q_3(\lambda) := \lambda - 9 - \frac{c^2}{\lambda-1} - \frac{c^2}{\lambda-25} + \frac{c^3}{(\lambda-1)^2} - \frac{c^4}{(\lambda-1)^3} - \frac{c^4}{(\lambda-25)^2(\lambda-49)} + \frac{c^5}{(\lambda-1)^4} \\ - \frac{c^6}{(\lambda-1)^5} - \frac{c^6}{(\lambda-25)^3(\lambda-49)^2} - \frac{c^6}{(\lambda-25)^2(\lambda-49)^2(\lambda-81)} + \frac{c^7}{(\lambda-1)^6} \\ = 0, \end{aligned} \tag{23}$$

and for $m = 3$ and $n = 4$ in (20),

$$\begin{aligned} Q_4(\lambda) := \lambda - 16 - \frac{c^2}{\lambda-4} - \frac{c^2}{\lambda-36} - \frac{c^4}{(\lambda-36)^2(\lambda-64)} \\ - \frac{c^6}{(\lambda-36)^3(\lambda-64)^2} - \frac{c^6}{(\lambda-36)^2(\lambda-64)^2(\lambda-100)} = 0. \end{aligned} \tag{24}$$

Then,

$$P_1(\lambda) := (\lambda-9)^3(\lambda-25)^2(\lambda-49)Q_1(\lambda), \tag{25}$$

$$P_2(\lambda) := (\lambda-16)^3(\lambda-36)^2(\lambda-64)Q_2(\lambda), \tag{26}$$

$$P_3(\lambda) := (\lambda-1)^6(\lambda-25)^3(\lambda-49)^2(\lambda-81)Q_3(\lambda) \tag{27}$$

and

$$P_4(\lambda) := (\lambda-4)(\lambda-36)^3(\lambda-64)^2(\lambda-100)Q_4(\lambda) \tag{28}$$

are polynomials of degree 7, 7, 13, and 8, respectively. By the same token, we can derive polynomials to approximate the other Dirichlet eigenvalues, for $n \geq 5$.

Now, we present a numerical example.

Example 1 For $m = 2$ and $c^2 = -2.157281295$, Veliev [19] approximated the first two periodic eigenvalues, say μ_0 and μ_2, which are also Neumann eigenvalues. Besides, we approximated [14] the first two antiperiodic eigenvalues, one of which is the first Dirichlet eigenvalue λ_1. We also approximated [14] the third periodic eigenvalue, which is the second Dirichlet eigenvalue λ_2. In this paper, we have obtained the same values for Dirichlet eigenvalues using completely different iteration formulas.

Now, we show that the first Dirichlet eigenvalue λ_1 is the complex eigenvalue lying inside the circle

$$C = \left\{\lambda \in \mathbb{C} : \left|\lambda - (1.26575008922 - 1.52020432568i)\right| = 1.7 \times 10^{-6}\right\}.$$

The root of the polynomial $P_1(\lambda)$ defined by (25), lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 2|c|\}$, is $r_1 = (1.26575008922 - 1.52020432568i)$. The other roots of $P_1(\lambda)$ are $r_2 = (8.96777697119 - 0.142338162679i)$, $r_3 = (8.79563202223 + 0.0317230792875i)$, $r_4 = (8.97007606112 + 0.162097407292i)$, $r_5 = (25.0005579806 + 0.00577397577187i)$, $r_6 = (25.0002071021 - 0.00582061314113i)$ and $r_7 = (48.9999997735 + 0.00000000692262634543i)$. Using the decomposition

$$Q_1(\lambda) = \frac{(\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_7)}{(\lambda - 9)^3(\lambda - 25)^2(\lambda - 49)},$$

by direct calculations, we obtain $|Q_1(\lambda)| > 4.6113 \times 10^{-6}$, for all $\lambda \in C$. On the other hand, one can easily calculate that $\sum_{k=1}^{\infty} |A_{2k+1}(\lambda)| < 4.4786 \times 10^{-6}$, for all $\lambda \in C$. The proof follows from Rouche's theorem and Theorem 1(a); equation (12) has only one root inside the circle C, and λ_1 is the complex eigenvalue lying inside C.

One can show in a similar way that the second Dirichlet eigenvalue λ_2 is the real eigenvalue lying inside the circle

$$D = \left\{\lambda \in \mathbb{C} : |\lambda - 4.1814942277| = 1.7 \times 10^{-6}\right\},$$

using Rouche's theorem and Theorem 2(a).

Similarly, we find the first 8 Dirichlet eigenvalues numerically for $c^2 = -2.157281295$ as follows:

$$\lambda_1 = 1.26575008922 - 1.52020432568i,$$

$$\lambda_2 = 4.1814942277,$$

$$\lambda_3 = 8.86899351832 + 0.0514847337328i,$$

$$\lambda_4 = 15.9263450168,$$

$$\lambda_5 = 24.9551222753 - 0.0000466505625109i,$$

$$\lambda_6 = 35.96920215,$$

$$\lambda_7 = 48.9775356736 + 0.00000000701304988868i,$$

$$\lambda_8 = 63.9828818845.$$

Moreover, separating the Dirichlet eigenvalues from the periodic and antiperiodic eigenvalues obtained in [14], we obtain the Neumann eigenvalues numerically. The first 9 Neumann eigenvalues for $c^2 = -2.157281295$ are as follows:

$$\mu_1 = 1.26575008922 + 1.52020432568i,$$

$$\mu_0 = 2.08869892467 - 0.000232839091042i,$$

$$\mu_2 = 2.08869892467 + 0.000232839091042i,$$

$$\mu_3 = 8.86899351832 - 0.0514847337328i,$$

$$\mu_4 = 15.9304406409,$$

$$\mu_5 = 24.9551222753 + 0.0000466505625109i,$$

$$\mu_6 = 35.9692007691,$$

$$\mu_7 = 48.9775356736 - 0.000000007013049888868i,$$

$$\mu_8 = 63.982881884.$$

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References

1. Bagarello, F., Gazeau, J.P., Szafraniec, F.H., Znojil, M. (eds.): Non-Selfadjoint Operators in Quantum Physics. Mathematical Aspects. Wiley, New York (2015)
2. Bender, C.M., Dunne, G.V., Meisinger, P.N.: Complex periodic potentials with real band spectra. Phys. Lett. A **252**, 272–276 (1999)
3. Brown, B.M., Eastham, M.S.P., Schmidt, K.M.: Periodic differential operators. In: Operator Theory: Advances and Applications, vol. 230. Birkhuser/Springer, Basel (2013)
4. Eastham, M.S.P.: The Spectral Theory of Periodic Differential Operators. Hafner, New York (1974)
5. Gasymov, M.G.: Spectral analysis of a class of second-order nonself-adjoint differential operators. Fankts. Anal. Prilozhen **14**, 14–19 (1980)
6. Kerimov, N.B.: On a boundary value problem of N. I. Ionkin type. Differ. Equ. **49**, 1233–1245 (2013)
7. Levy, M., Keller, B.: Instability intervals of Hill's equation. Commun. Pure Appl. Math. **16**, 469–476 (1963)
8. Magnus, W., Winkler, S.: Hill's Equation. Interscience Publishers, New York (1966)
9. Makris, K.G., El-Ganainy, R., Christodoulides, D.N., Musslimani, Z.H.: PT-symmetric optical lattices. Phys. Rev. A **81**, 063807 (2010)
10. Makris, K.G., El-Ganainy, R., Christodoulides, D.N., Musslimani, Z.H.: PT-symmetric periodic optical potentials. Int. J. Theor. Phys. **50**, 1019–1041 (2011)

11. Marchenko, V.: Sturm–Liouville Operators and Applications. Birkhauser Verlag, Basel (1986)
12. Midya, B., Roy, B., Roychoudhury, R.: A note on the PT invariant periodic potential $4\cos^2 x + 4iV \sin 2x$. Phys. Lett. A **374**, 2605–2607 (2010)
13. Mostafazadeh, A.: Pseudo-Hermitian representation of quantum mechanics. Int. J. Geom. Methods Mod. Phys. **11**, 1191–1306 (2010)
14. Nur, C.: Computing Periodic and Antiperiodic Eigenvalues with a PT-Symmetric Optical Potential. Math. Notes (2023, in press)
15. Veliev, O.A.: Isospectral Mathieu–Hill operators. Lett. Math. Phys. **103**, 919–925 (2013)
16. Veliev, O.A.: On the simplicity of the eigenvalues of the non-self-adjoint Mathieu–Hill operators. Appl. Comput. Math. **13**, 122–134 (2014)
17. Veliev, O.A.: Spectral problems of a class of non-self-adjoint one-dimensional Schrödinger operators. J. Math. Anal. Appl. **422**, 1390–1401 (2015)
18. Veliev, O.A.: On the spectral properties of the Schrodinger operator with a periodic PT-symmetric potential. Int. J. Geom. Methods Mod. Phys. **14**, 1750065 (2017)
19. Veliev, O.A.: The spectrum of the Hamiltonian with a PT-symmetric periodic optical potential. Int. J. Geom. Methods Mod. Phys. **15**, 1850008 (2018)
20. Veliev, O.A.: Spectral analysis of the Schrödinger operator with a PT-symmetric periodic optical potential. J. Math. Phys. **61**, 063508 (2020)
21. Veliev, O.A.: Non-self-Adjoint Schrödinger Operator with a Periodic Potential. Springer, Cham (2021)
22. Yilmaz, B., Veliev, O.A.: Asymptotic formulas for Dirichlet boundary value problems. Studia Sci. Math. Hung. **42**(2), 153–171 (2005)

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