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COMPARATIVE ANALYSIS OF THE GENERALIZED UNIFIED METHOD WITH SOME EXACT SOLUTION METHODS AND GENERAL SOLUTIONS OF THE BISWAS–MILOVIC EQUATION

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The aim of this study is twofold. First, we compare the generalized unified method (GUM), which is a new expansion method to solve nonlinear partial differential equations (NPDEs), with some methods frequently used for finding exact solutions of NPDEs. We conclude that the GUM gives more general solutions efficiently, in compact form, and with free parameters. Moreover, the algorithm of the GUM is straightforward and easy to implement on a computer. Second, as a practical example and a demonstration of effectiveness, we apply the GUM to the Biswas–Milovic equation (BME). The BME is derived from a generalized nonlinear Schrödinger equation. The BME appears in many applied fields such as the propagation of waves in nonlinear optics. We consider Kerr, power, parabolic, and dual-power-law nonlinearities of the BME. Using the GUM, we obtain the exact solution of the BME in an elegant way.

Keywords: generalized unified method, unified method, Biswas–Milovic equation with Kerr, power, parabolic, and dual-power-law nonlinearities, exact solution method

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1. Introduction

Nonlinear partial differential equations (NPDEs) have emerged in many applied fields of science, particularly physics, mathematics, and engineering problems as they turn real-world phenomena into physical models governed by mathematics. Therefore, there is increasing attention to methods that provide exact solutions of NPDEs. In particular, more straightforward and more direct new solution methods for the NPDEs are required. In the past several decades, many powerful methods have been proposed and employed to obtain these solutions, such as the homogeneous balance method introduced by Wang [1], the $\left(\frac{G'}{G}\right)$-expansion method introduced by Wang and Zhang [2], the tanh-function method introduced by Malfliet [3], and the unified method [4], [5]. By all means, these methods work well in finding exact solutions, but have some limitations, such as a high computational complexity or providing the same solution in different forms. Besides, it is necessary in some cases to use more than one method to obtain more and different types of solutions. In our previous studies, we comparatively examined the unified method, the tanh method family, the $\left(\frac{G'}{G}\right)$-expansion method family, and the extended homogeneous balance method [4]–[7]. Among all these methods, the unified method has provided many solutions, without the need for different methods to find more solutions of NPDEs [8]–[17]. We then proposed the generalized unified method (GUM) [18] with complex arbitrary constants to solve NPDEs more compactly.

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The aim of this study is twofold. First, we describe the steps of the GUM algorithm apply it to the Biswas–Milovic equation (BME) with Kerr, power, parabolic, and dual-power law nonlinearities as an illustrative example. Second, we prove that compared to other methods mostly used to solve NPDEs, the GUM provides general and more solutions in compact form with free parameters.

The reason for using the BME as an illustrative example is the relation between the BME and the generalized nonlinear Schrödinger equation (NLSE). The NLSE plays a key role in characterizing novel properties of complex phenomena in diverse areas of applied mathematics and physics such as nonlinear optics, nonlinear acoustics, plasma physics, telecommunications, hydrodynamics, fluid dynamics, water waves, wave processes in optical fibers, and the dynamics of particles. Most of the problems that originate from these scientific areas are shaped by the nonlinearity term of the NLSE.

The BME that describes the propagation of wave packets through optical fibers is given as

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta F(|u|^2)u^m = 0, \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function with two independent variables x (spatial) and t (temporal). Here, α and β are real constants, and $m \geqslant 1$ converts the NLSE to the BME. The parameter m indicates the generalization degree of the NLSE to the BME, and the BME is reduced to the generalized NLSE when $m = 1$. It is worth noting that obtaining the exact traveling wave solutions of the BME means obtaining the exact traveling wave solutions of the generalized NLSE.

Originally, Biswas and Milovic proposed four types of nonlinearity: Kerr, power, parabolic, and dual-power law ones [19].

$$F(w) = \begin{cases} w, & \text{Kerr law nonlinearity,} \\ w^n, & \text{power-law nonlinearity,} \\ w + \gamma w^2, & \text{parabolic-law nonlinearity,} \\ w^n + \gamma w^{2n}, & \text{dual-power-law nonlinearity.} \end{cases} \tag{1.2}$$

Some of the problems in optics corresponding to these nonlinearities are briefly as follows. Kerr's law nonlinearity arises when a light wave in an optical fiber encounters nonlinear responses resulting from the nonharmonic motion of electrons bound in molecules caused by an external electric field. Power-law nonlinearity arises in plasma physics and nonlinear fiber optics [20]. Parabolic-law nonlinearity arises in describing the nonlinear interaction between the high-frequency Langmuir waves and the ion–acoustic waves. Dual-power law nonlinearity arises in describing the solitons in photovoltaic/photorefractive materials and the saturation of the nonlinear refractive index.

Both topological (dark) and nontopological (bright) solitons have been obtained for BME (1.1) with four nonlinearities, first introduced by Biswas and Milovic [19]. Thereafter, the BME has received great attention due to its versatile applications in many scientific fields mentioned above, and the exact and numeric solutions of the BME have been studied by many researchers. Khalique [21] use the Lie symmetry analysis to find solutions of the BME. Jafari et al. [22] obtained dark solitons solutions of the BME by using the first integral method. An extended $\left(\frac{G'}{G}\right)$-expansion method and the first integral method were used to solve the BME by Zhou et al. [23], [24]. The exact solutions of BME with four nonlinearities are obtained by Raza et al. [25] using the $\phi(\xi)$-expansion function method. Tahir and Awan [26] worked on the BME with power-law and dual-power-law nonlinearities to construct singular and dark one-soliton solutions by implementing the first integral method. Rizvi et al. [27] applied the new extended auxiliary equation method to obtain bright, dark, singular, and other solitary wave solutions of the BME with the Kerr law and dual-power-law nonlinearities. Ozisik [28] showed that the exact solutions of the BME equation with Kerr law nonlinearity in $(2+1)$ and $(3+1)$ dimensions are obtained by using the new Kudryashov method and the unified Riccati equation expansion method. The BME with Kudryashov's law and nonlinear perturbation terms in polarization-preserving fibers was studied by Akinyemi et al. [29] using the first integral method. Akinyemi et al. [30] applied three different methods (the simple equation method, the $\left(\frac{G'}{G}\right)$-expansion method, and the new Kudryashov method) to find the exact solitary wave solutions of the perturbed BME with Kudryashov's law for the refractive index. The modified Kudryashov's algorithm and the addendum to Kudryashov's approach are used to find solutions for BME with dual-power-law nonlinearity by Zayed et al. [31]. Ozisik et al. [32] derived exact solutions of the BME with Kerr law nonlinearity using five different new extended auxiliary equation approaches. The time-fractional BME with Kerr law nonlinearity was considered by Gupta and Yadav [33] by applying the new extended direct algebraic method. The BME with Kerr law nonlinearity was investigated numerically by Ahmed et al. [34] using the modified Adomian decomposition method.

This paper is organized as follows. In Sec. 2, a brief description of the GUM is given. The application of the GUM to the BME is presented in Sec. 3. The solutions obtained by the GUM are discussed in Sec. 4. Concluding remarks are given in Sec. 5.

2. The algorithm of the generalized unified method

In this section, we list the steps of the GUM for solving NPDEs. Let a general form NPDE for an unknown function $u = u(x,t)$ of two independent variables x and t be defined by a function F that depends on the higher-order derivatives and terms nonlinear in u:

$$F(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0. \tag{2.1}$$

Step 1. By applying the wave transformation $\eta = x - ct + \eta_0$ to the unknown function $u(x,t) = U(\eta)$, we convert the NPDE in Eq. (2.1) into a nonlinear ordinary differential equation (NODE) as follows:

$$F(U, U', U'', U''', \ldots) = 0. \tag{2.2}$$

Step 2. The solution of Eq. (2.2) can be expressed as an ansatz

$$U(\eta) = a_0 + \sum_{m=1}^{M} [a_m \phi^m + b_m \phi^{-m}], \tag{2.3}$$

where a_0, a_m, and b_m are to be determined later, and $\phi = \phi(\eta)$ satisfies the Riccati differential equation defined below with $\phi' = d\phi/d\eta$ and $\mu = d + if$ such that d and f are parameters:

$$\phi'(\eta) = \phi^2(\eta) - \mu^2. \tag{2.4}$$

The general solutions of Eq. (2.4) are given by

$$\phi(\eta) = \begin{cases} \phi_1 = \dfrac{(d+if)\sqrt{A^2 + (B+iC)^2} - A(d+if)\cosh(2(d+if)(\eta+\eta_0))}{(B+iC) + A\sinh(2(d+if)(\eta+\eta_0))}, \\ \phi_2 = \dfrac{-(d+if)\sqrt{A^2 + (B+iC)^2} - A(d+if)\cosh(2(d+if)(\eta+\eta_0))}{(B+iC) + A\sinh(2(d+if)(\eta+\eta_0))}, \\ \phi_3 = \dfrac{(d+if)(-A + e^{-2(d+if)(\eta+\eta_0)})}{A + e^{-2(d+if)(\eta+\eta_0)}}, \\ \phi_4 = -\dfrac{(d+if)(-A + e^{2(d+if)(\eta+\eta_0)})}{A + e^{2(d+if)(\eta+\eta_0)}}, \\ \phi_5 = -\dfrac{1}{\eta+\eta_0}, \end{cases} \tag{2.5}$$

where $A \neq 0$, B, C, and η_0 are real arbitrary parameters.

Step 3. Considering the balance between the linear term of the highest order with the nonlinear term of the highest degree, the balance parameter M in Eq. (2.3) can be determined.

Step 4. U represented by a finite series expansion in Eq. (2.3) and its derivatives are substituted in Eq. (2.2) to obtain a set of algebraic equations that gives the traveling wave solutions of Eq. (2.1). Taking Eq. (2.4) into consideration when calculating the derivatives of U simplifies the solution process on the computer.

Step 5. Solving the obtained algebraic systems and substituting the solution sets in Eq. (2.3) gives the traveling wave solutions of Eq. (2.1) in closed form with free parameters A, B, and C.

3. The exact solutions of the Biswas–Milovic equation by the GUM

The BME, as a dimensionless mathematical form of the generalized NLSE, is given by

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta F(|u|^2)u^m = 0, \tag{3.1}$$

where $m \geqslant 1$, α, and β are real constants. In this study, we investigate exact solutions of Eq. (3.1) for various types of the F functions in (1.2).

To apply the GUM, we first reduce the NPDE to a NODE by using the wave transformation $u(x,t) = U(\eta)e^{-i\Omega}$. Because $u(x,t)$ is a complex-valued function, the solution of the BME is represented in a phase–amplitude form as $u(x,t) = U(\eta)e^{-i\Omega}$. Here, $U(\eta)$ with the wave variable $\eta = x - ct + \eta_0$ defines the pulse shape, where c is the velocity, and $\Omega = px - vt + \Omega_0$ defines the phase, where p and v are the frequency and wave number, and Ω_0 and η_0 are arbitrary free parameters. The shape of the amplitude of the wave depends on the nonlinear function F in Eq. (1.2).

Substituting $u(x,t) = U(\eta)e^{-i\Omega}$ and its derivatives in Eq. (3.1),

$$\begin{aligned}
&(u^m(x,t))_t = (U^m(\eta)e^{-im\Omega})_t = [imvU^m - cmU^{m-1}U']e^{-im\Omega},\\
&(u^m(x,t))_x = (U^m(\eta)e^{-im\Omega})_x = [-impU^m + mU^{m-1}U']e^{-im\Omega},\\
&(u^m(x,t))_{xx} = (U^m(\eta)e^{-im\Omega})_{xx} = [-2im^2pU^{m-1}U' - m^2p^2U^m + m(m-1)U^{m-2}U'^2 +\\
&\qquad\qquad + mU^{m-1}U'']e^{-im\Omega},
\end{aligned} \tag{3.2}$$

we obtain the following NODEs for the imaginary and real parts:

$$-cmU^{m-1}U' - 2\alpha m^2pU^{m-1}U' = 0, \tag{3.3}$$

and

$$-(mv + \alpha m^2p^2)U^m + \alpha m(m-1)U^{m-2}U'^2 + \alpha mU^{m-1}U'' + \beta F(U^2)U^m = 0. \tag{3.4}$$

Equation (3.3) provides the velocity of the wave as $c = -2\alpha mp$. In what follows, we analyze Eq. (3.4) to find exact solutions of the BME by applying the GUM for each nonlinearity.

3.1. The Biswas–Milovic equation with Kerr law nonlinearity. For the Kerr law nonlinearity, BME (3.1) takes the form

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta(|u|^2)u^m = 0. \tag{3.5}$$

After the wave transformation of the last term, Eq. (3.4) becomes

$$-(mv + \alpha m^2p^2)U^2 + \alpha m(m-1)U'^2 + \alpha mUU'' + \beta U^4 = 0. \tag{3.6}$$

Balancing the term U'^2 with the nonlinear term U^4 gives this simple equation $4M = 2M + 2$. With the balance parameter M, the solution of Eq. (3.6) is defined as

$$U(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}, \tag{3.7}$$

where a_0, a_1, and b_1 are coefficients of ϕ to be determined later. Substituting Eq. (3.7) and its derivatives in Eq. (3.6) and equating each coefficient of ϕ to zero, we obtain a system of algebraic equations for a_0, a_1, b_1, and p. Solving this system with `Maple`, we obtain

$$a_0 = 0, \qquad a_1 = \mp\frac{\sqrt{-\beta(m+1)\alpha m}}{\beta},$$
$$b_1 = \frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}, \qquad p = \mp\frac{\sqrt{\alpha m(4\alpha m\mu^2 - v)}}{\alpha m}.$$

Combining the general solutions (2.5) with the solution set of the system of algebraic equations gives the general solution of the BME with Kerr law nonlinearity:

$$u_{\text{Kerr}}(x,t) = \left(\mp\frac{\sqrt{-\beta(m+1)\alpha m}}{\beta}\phi + \frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}\phi}\right)e^{i\left(\pm\frac{\sqrt{\alpha m(4\alpha m\mu^2-v)}}{\alpha m}x+vt-\Omega_0\right)}. \tag{3.8}$$

3.2. The Biswas–Milovic equation with power-law nonlinearity. With the power-law nonlinearity, BME (3.1) takes the form

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta(|u|^{2n})u^m = 0. \tag{3.9}$$

After the wave transformation of the last term, Eq. (3.4) becomes

$$-(mv + \alpha m^2p^2)U^2 + \alpha m(m-1)U'^2 + \alpha mUU'' + \beta U^{2n+2} = 0. \tag{3.10}$$

Balancing the term U'^2 with the nonlinear term U^{2n+2} gives the simple equation $(2n+2)M = 2M + 2$. To obtain a convenient balance number, we use the transformation $U(\eta) = W^{1/n}(\eta)$ in Eq. (3.10):

$$-(mv + \alpha m^2p^2)W^2 + \frac{\alpha m(m-n)}{n^2}W'^2 + \frac{\alpha m}{n}WW'' + \beta W^4 = 0. \tag{3.11}$$

Balancing the term W'^2 with the nonlinear term W^4 gives the simple equation $4M = 2M + 2$. With the balance parameter M, the solution of Eq. (3.11) is defined as

$$W(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}, \tag{3.12}$$

where a_0, a_1, and b_1 are coefficients of ϕ to be determined later. Substituting Eq. (3.12) and its derivatives in Eq. (3.11) and equating each coefficient of ϕ to zero, we obtain a system of algebraic equations for a_0, a_1, b_1, and p. Solving this system with `Maple`, we obtain

$$a_0 = 0, \qquad a_1 = \mp\frac{\sqrt{-\beta(m+n)\alpha m}}{\beta n},$$
$$b_1 = \frac{\alpha m\mu^2(m+n)}{n\sqrt{-\beta\alpha m(m+n)}}, \qquad p = \mp\frac{\sqrt{\alpha m(4\alpha m\mu^2 - n^2v)}}{\alpha mn}.$$

Combining the general solutions in (2.5) with the solution set of the system of algebraic equations gives the general solution of the BME with power law nonlinearity:

$$u_{\text{power}}(x,t) = \left(\mp\frac{\sqrt{-\beta(m+n)\alpha m}}{\beta n}\phi + \frac{\alpha m\mu^2(m+n)}{\sqrt{-\beta\alpha m(m+n)}\phi}\right)^{1/n} e^{i\left(\pm\frac{\sqrt{\alpha m(4\alpha m\mu^2-n^2v)}}{\alpha mn}x+vt-\Omega_0\right)}. \tag{3.13}$$

3.3. The Biswas–Milovic equation with parabolic-law nonlinearity. With the parabolic-law nonlinearity, BME (3.1) takes the form

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta(|u|^2 + \gamma|u|^4)u^m = 0. \tag{3.14}$$

After the wave transformation of the last term, Eq. (3.4) becomes

$$-(mv + \alpha m^2p^2)U^2 + \alpha m(m-1)U'^2 + \alpha mUU'' + \beta(U^4 + \gamma U^6) = 0. \tag{3.15}$$

Balancing the term U'^2 with the nonlinear term U^6 gives the simple equation $6M = 2M + 2$. To obtain a convenient balance number for Eq. (3.15), we use the transformation $U(\eta) = W^{1/2}(\eta)$ in Eq. (3.15)

$$-(mv + \alpha m^2p^2)W^2 + \frac{\alpha m(m-2)}{4}W'^2 + \frac{\alpha m}{2}WW'' + \beta(W^3 + \gamma W^4) = 0. \tag{3.16}$$

Balancing the term W'^2 with the nonlinear term W^4 gives the simple equation $4M = 2M + 2$. With the balance parameter M, the solution of Eq. (3.16) is defined as

$$W(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}. \tag{3.17}$$

Substituting Eq. (3.17) and its derivatives in Eq. (3.16) and equating each coefficient of ϕ to zero, we obtain a system of algebraic equations for a_0, a_1, b_1, μ, and v. Solving this system with `Maple`, we obtain the following sets of coefficients.

Set 1:

$$a_0 = -\frac{m+2}{4\gamma(m+1)}, \qquad a_1 = 0, \qquad b_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}\mu^2}{2\gamma\beta},$$
$$\mu = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{2\gamma\alpha m(m+1)}, \qquad v = -\frac{4\gamma\alpha m^3p^2 + 8\gamma\alpha m^2p^2 + \beta m + 4\gamma\alpha mp^2 + 2\beta}{4\gamma(m^2+2m+1)}.$$

Set 2:

$$a_0 = -\frac{m+2}{4\gamma(m+1)}, \qquad a_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{2\gamma\beta}, \qquad b_1 = 0,$$
$$\mu = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{2\gamma\alpha m(m+1)}, \qquad v = -\frac{4\gamma\alpha m^3p^2 + 8\gamma\alpha m^2p^2 + \beta m + 4\gamma\alpha mp^2 + 2\beta}{4\gamma(m^2+2m+1)}.$$

Set 3:

$$a_0 = -\frac{m+2}{4\gamma(m+1)}, \quad a_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{2\gamma\beta}, \quad b_1 = \pm\frac{(m+2)\sqrt{-\beta\gamma\alpha m(m+2)}}{32\gamma^2\alpha m(m^2+2m+1)},$$
$$\mu = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{4\gamma\alpha m(m+1)}, \qquad v = -\frac{4\gamma\alpha m^3p^2 + 8\gamma\alpha m^2p^2 + \beta m + 4\gamma\alpha mp^2 + 2\beta}{4\gamma(m^2+2m+1)}.$$

Combining the general solutions in (2.5) with the solution sets of the system of algebraic equations gives the general solutions of the BME with parabolic law nonlinearity in the form

$$u_{\text{parabolic}_1}(x,t) = \left(-\frac{m+2}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2)}\mu^2}{2\gamma\beta\phi}\right)^{1/2} e^{-i(px - vt + \Omega_0)}, \tag{3.18}$$

$$u_{\text{parabolic}_2}(x,t) = \left(-\frac{m+2}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2)}\phi}{2\gamma\beta}\right)^{1/2} e^{-i(px - vt + \Omega_0)}, \tag{3.19}$$

where

$$\mu = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{2\gamma\alpha m(m+1)}, \qquad v = -\frac{4\gamma\alpha m^3p^2 + 8\gamma\alpha m^2p^2 + \beta m + 4\gamma\alpha mp^2 + 2\beta}{4\gamma(m^2+2m+1)},$$

and

$$u_{\text{parabolic}_3}(x,t) = \left(-\frac{m+2}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2)}\phi}{2\gamma\beta} \mp \frac{(m+2)\sqrt{-\beta\gamma\alpha m(m+2)}}{32\gamma^2\alpha m(m^2+2m+1)\phi}\right)^{1/2} \times \times e^{-i(px-vt+\Omega_0)}, \tag{3.20}$$

where

$$\mu = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2)}}{4\gamma\alpha m(m+1)}, \qquad v = -\frac{4\gamma\alpha m^3p^2 + 8\gamma\alpha m^2p^2 + \beta m + 4\gamma\alpha mp^2 + 2\beta}{4\gamma(m^2+2m+1)}.$$

3.4. The Biswas–Milovic equation with dual-power law nonlinearity. With the dual-power-law nonlinearity, BME (3.1) takes the form

$$i(u^m)_t + \alpha(u^m)_{xx} + \beta(|u|^{2n} + \gamma|u|^{4n})u^m = 0. \tag{3.21}$$

After the wave transformation of the last term, Eq. (3.4) becomes

$$-(mv + \alpha m^2p^2)U^2 + \alpha m(m-1)U'^2 + \alpha mUU'' + \beta(U^{2n+2} + \gamma U^{4n+2}) = 0. \tag{3.22}$$

Balancing the term U'^2 with the nonlinear term U^{4n+2} gives the simple equation $(4n+2)M = 2M+2$. To obtain a convenient balance number, we use the transformation $U(\eta) = W^{1/2n}(\eta)$ in Eq. (3.22):

$$-(mv + \alpha m^2p^2)W^2 + \frac{\alpha m(m-2n)}{4n^2}W'^2 + \frac{\alpha m}{2n}WW'' + \beta(W^3 + \gamma W^4) = 0. \tag{3.23}$$

Balancing the term W'^2 with the nonlinear term W^4 gives the simple equation $4M = 2M+2$. With the balance parameter M, the solution of Eq. (3.23) is defined as

$$W(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}. \tag{3.24}$$

Substituting Eq. (3.24) and its derivatives in Eq. (3.23) and equating each coefficient of ϕ to zero, we obtain a system of algebraic equations for a_0, a_1, b_1, μ, and v. Solving this system with `Maple`, we obtain the following sets of coefficients.

Set 1:

$$a_0 = -\frac{m+2n}{4\gamma(m+n)}, \qquad a_1 = 0, \qquad b_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}(m+2n)}{8\gamma^2\alpha m(m+n)^2},$$

$$\mu = \pm\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{2\gamma\alpha m(m+n)},$$

$$v = -\frac{4\alpha mp^2\gamma n^2 + 8n\alpha m^2p^2\gamma + 2\beta n + \beta m + 4\gamma\alpha m^3p^2}{4\gamma(n^2+2nm+m^2)}.$$

Set 2:

$$a_0 = -\frac{m+2n}{4\gamma(m+n)}, \qquad a_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{2\gamma\beta n}, \qquad b_1 = 0,$$

$$\mu = \pm\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{2\gamma\alpha m(m+n)},$$

$$v = -\frac{4\alpha mp^2\gamma n^2 + 8n\alpha m^2p^2\gamma + 2\beta n + \beta m + 4\gamma\alpha m^3p^2}{4\gamma(n^2+2nm+m^2)}.$$

Set 3:

$$a_0 = -\frac{m+2n}{4\gamma(m+n)}, \qquad a_1 = \mp\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{2\gamma\beta n},$$
$$b_1 = \pm\frac{n(m+2n)\sqrt{-\beta\gamma\alpha m(m+2n)}}{32\gamma^2\alpha m(m^2+2mn+n^2)},$$
$$\mu = \pm\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{4\gamma\alpha m(m+n)},$$
$$v = -\frac{4\alpha mp^2\gamma n^2 + 8n\alpha m^2p^2\gamma + 2\beta n + \beta m + 4\gamma\alpha m^3p^2}{4\gamma(n^2+2nm+m^2)}.$$

Combining the general solutions in (2.5) with the solution sets of the system of algebraic equations gives the general solutions of the BME with dual-power law nonlinearity in the form

$$u_{\mathrm{dual}_1}(x,t) = \left(-\frac{m+2n}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2)}(m+2n)}{8\gamma^2\alpha m(m+n)^2\phi}\right)^{1/2n} e^{-i(px-vt+\Omega_0)}, \tag{3.25}$$
$$u_{\mathrm{dual}_2}(x,t) = \left(-\frac{m+2n}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2)}\phi}{2\gamma\beta n}\right)^{1/2n} e^{-i(px-vt+\Omega_0)}, \tag{3.26}$$

where

$$\mu = \pm\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{2\gamma\alpha m(m+n)}, \qquad v = -\frac{4\alpha mp^2\gamma n^2 + 8n\alpha m^2p^2\gamma + 2\beta n + \beta m + 4\gamma\alpha m^3p^2}{4\gamma(n^2+2nm+m^2)},$$

and

$$u_{\mathrm{dual}_3}(x,t) = \left(-\frac{m+2n}{4\gamma(m+1)} \mp \frac{\sqrt{-\beta\gamma\alpha m(m+2n)}\phi}{2\gamma\beta n} \mp \frac{n(m+2n)\sqrt{-\beta\gamma\alpha m(m+2n)}}{32\gamma^2\alpha m(m^2+2mn+n^2)\phi}\right)^{1/2} \times \times e^{-i(px-vt+\Omega_0)}, \tag{3.27}$$

where

$$\mu = \pm\frac{\sqrt{-\beta\gamma\alpha m(m+2n)}}{4\gamma\alpha m(m+n)}, \qquad v = -\frac{4\alpha mp^2\gamma n^2 + 8n\alpha m^2p^2\gamma + 2\beta n + \beta m + 4\gamma\alpha m^3p^2}{4\gamma(n^2+2nm+m^2)}.$$

4. Results and discussion

Applying the GUM, we have derived more general forms of exact solutions of the BME with four nonlinearity types. Combining the general solutions in (2.5) with each solution set provides at least four different general exact solutions. These general solutions can easily be converted from hyperbolic to trigonometric solutions or vice versa with the aid of the hyperbolic–trigonometric identities $\sinh(ix) = i\sin x$ and $\cosh(ix) = \cos x$. The rational solution sometimes does not exist because conditions on the coefficients reduce the solution to a trivial one.

In this section, we explain in more detail how a specific solution with the Kerr nonlinearity can be obtained algebraically from the above set of solutions. Solutions for the other nonlinearities in (3.13), (3.18), (3.19), and (3.23)–(3.25) can be derived similarly.

From the GUM solution to the unified method solutions. The hyperbolic and trigonometric solutions of the unified method are derived from the general solution of the BME with Kerr nonlinearity in (3.8). After setting $B = 0$ and $C = 0$ in (2.5), the exact solutions take the form

$$u_{k1}(x,t) = \left(\frac{\mp \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta}(\mp\sqrt{A^2+B^2} - A\cosh(2\mu(x-ct+\eta_0)))}{B \mp A\sinh(2\mu(x-ct+\eta_0))} + \right.$$
$$\left. + \frac{\frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}(B \mp A\sinh(2\mu(x-ct+\eta_0)))}{\mp\sqrt{A^2+B^2} - A\cosh(2\mu(x-ct+\eta_0))} \right) e^{-i\Omega},$$

$$u_{k2}(x,t) = \left(\frac{\mp \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta}(\mp\sqrt{A^2-C^2} - A\cosh(2\mu(x-ct+\eta_0)))}{iC \mp A\sinh(2\mu(x-ct+\eta_0))} + \right.$$
$$\left. + \frac{\frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}(iC \mp A\sinh(2\mu(x-ct+\eta_0)))}{\mp\sqrt{A^2-C^2} - A\cosh(2\mu(x-ct+\eta_0))} \right) e^{-i\Omega},$$

$$u_{k3}(x,t) = \left(\frac{\mp \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta}(-A + e^{2\mu(x-ct+\eta_0)})}{A + e^{2\mu(x-ct+\eta_0)}} + \right.$$
$$\left. + \frac{\frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}(A + e^{2\mu(x-ct+\eta_0)})}{-A + e^{2\mu(x-ct+\eta_0)}} \right) e^{-i\Omega},$$

$$u_{k4}(x,t) = \left(\frac{\mp \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta}(-A + e^{-2\mu(x-ct+\eta_0)})}{A + e^{-2\mu(x-ct+\eta_0)}} + \right.$$
$$\left. + \frac{\frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}(A + e^{-2\mu(x-ct+\eta_0)})}{-A + e^{-2\mu(x-ct+\eta_0)}} \right) e^{-i\Omega},$$

where

$$\Omega = \mp \frac{\sqrt{\alpha m(4\alpha m\mu^2 - v)}}{\alpha m} x - vt + \Omega_0, \qquad c = -2\sqrt{\alpha m(4\alpha m\mu^2 - v)}.$$

The solutions $u_{k1}(x,t)$ and $u_{k4}(x,t)$ give exactly the hyperbolic solutions of the unified method. Depending on the complexity of μ, the trigonometric solutions of the unified method can be derived easily from these solution sets.

From the GUM solution to the tanh method solutions. Using hyperbolic–trigonometric equalities $\cosh(2a) = 2\cosh^2(a) - 1 = 2\sinh^2(a) + 1$ and $\sinh(2a) = 2\sinh(a)\cosh(a)$, the solutions obtained by the tanh method can be reproduced by setting $B = 0$ in $u_{k1}(x,t)$ and $C = 0$ in $u_{k2}(x,t)$:

$$u_{k5}(x,t) = \left(\pm \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta} \tanh(\mu(x-ct+\eta_0)) - \right.$$
$$\left. - \frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}} \coth(\mu(x-ct+\eta_0)) \right) e^{-i\Omega},$$
$$u_{k6}(x,t) = \left(\pm \frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta} \coth(\mu(x-ct+\eta_0)) - \right.$$
$$\left. - \frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}} \tanh(\mu(x-ct+\eta_0)) \right) e^{-i\Omega}.$$

where Ω and c are the same as in the preceding case. Using the identity $\tanh(a) = i\coth\left(\frac{\pi}{2}i + a\right)$, this solution of the tanh type can easily be converted into a coth-type solution. We can obtain the trigonometric solution in a similar manner when μ is a purely imaginary number.

From the GUM solution to the $\left(\frac{G'}{G}\right)$-expansion method solutions. Using the hyperbolic identities

$$\cosh(a \mp b) = \cosh(a)\cosh(b) \mp \sinh(a)\sinh(b), \qquad \sinh(a \mp b) = \sinh(a)\cosh(b) \mp \cosh(a)\sinh(b),$$

the solutions obtained by the $\left(\frac{G'}{G}\right)$-expansion method can be reproduced by transforming the function $\tanh(\mu(x-ct)+\mu\eta_0))$ in the solution $u_{k5}(x,t)$ with $a = \mu(x-ct)$ and $b = \mu\eta_0$:

$$u_{k7}(x,t) = \left(\mp\frac{\sqrt{-\beta(m+1)\alpha m}\mu}{\beta}\frac{C_1\sinh(\mu(x-ct)) + C_2\cosh(\mu(x-ct))}{C_1\cosh(\mu(x-ct)) + C_2\sinh(\mu(x-ct))} + \right.$$
$$\left. + \frac{\alpha m\mu^2(m+1)}{\sqrt{-\beta\alpha m(m+1)}}\frac{C_1\cosh(\mu(x-ct)) + C_2\sinh(\mu(x-ct))}{C_1\sinh(\mu(x-ct)) + C_2\cosh(\mu(x-ct))}\right)e^{-i\Omega},$$

where $C_1 = \cosh(\mu\eta_0)$, $C_2 = \sinh(\mu\eta_0)$, and $c = -2\sqrt{\alpha m(4\alpha m\mu^2 - v)}$. The trigonometric solution can be obtained when μ is a purely imaginary number.

5. Conclusions

The main results in this paper are as follows.

- We have compared the GUM with some methods of exact solution that are mostly used to solve NLPDEs. As a result, we have proved that the GUM gives many solutions in compact form with free parameters $A \neq 0$, B, C, and η_0. Therefore, the solutions can be obtained using the free parameters without reproducing the same solution in different forms and are easily converted from trigonometric to hyperbolic form or vice versa.
- The algorithm is very simple and very easy to perform on the computer, compared with other methods. In other words, more solutions are obtained effortlessly without doing tedious calculations.
- We have successfully implemented the GUM to construct exact solutions with free parameters for the BME with Kerr, power, parabolic, and dual-power-law nonlinearities. Therefore, this new computational method gives more general solutions to efficiently solve the problems modeled by NLPDEs in physics, mathematics, engineering, and optics.
- The GUM can be applied not only to NPDEs with two independent variables but also to NPDEs with more than two independent variables. In that case, the wave variable can be modified to $\eta = k_1x_1 + k_2x_2 + \cdots + k_nx_n - vt$ to find the unknown $k_1, k_2, \ldots, k_n$ in the solution process. Moreover, the GUM also implemented fractional NPDEs with more than two independent variables.

The computations in this work have been performed with `Maple` 12.

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