Pramana – J. Phys. (2018) 90:78
https://doi.org/10.1007/s12043-018-1564-7

Propagation of nonlinear shock waves for the generalised Oskolkov equation and its dynamic motions in the presence of an external periodic perturbation

TURGUT AK[1,*], TUGBA AYDEMIR[2], ASIT SAHA[3] and ABDUL HAMID KARA[4]

[1]Department of Transportation Engineering, Yalova University, 77100 Yalova, Turkey
[2]Yalova University, 77100 Yalova, Turkey
[3]Department of Mathematics, Sikkim Manipal Institute of Technology, Sikkim Manipal University, Majitar, Rangpo 737 136, India
[4]School of Mathematics, University of the Witwatersrand, Johannesburg 2050, South Africa
*Corresponding author. E-mail: akturgut@yahoo.com

MS received 22 July 2017; revised 16 November 2017; accepted 9 December 2017; published online 16 May 2018

Abstract. Propagation of nonlinear shock waves for the generalised Oskolkov equation and dynamic motions of the perturbed Oskolkov equation are investigated. Employing the unified method, a collection of exact shock wave solutions for the generalised Oskolkov equations is presented. Collocation finite element method is applied to the generalised Oskolkov equation for checking the accuracy of the proposed method by two test problems including the motion of shock wave and evolution of waves with Gaussian and undular bore initial conditions. Considering an external periodic perturbation, the dynamic motions of the perturbed generalised Oskolkov equation are studied depending on the system parameters with the help of phase portrait and time series plot. The perturbed generalised Oskolkov equation exhibits period-3, quasiperiodic and chaotic motions for some special values of the system parameters, whereas the generalised Oskolkov equation presents shock waves in the absence of external periodic perturbation.

Keywords. Generalised Oskolkov equation; shock wave; unified method; collocation; quasiperiodicity; chaos.

PACS NOs. **12.60.Jv; 12.10.Dm; 98.80.Cq; 11.30.Hv**

1. Introduction

During the last few decades, the study of nonlinear evolution equations (NEEs) has received much attention as these equations are used to model many physical problems in diverse fields of nonlinear science, such as fluid mechanics, nuclear physics, solid-state physics, plasma physics, chemical physics, optical fibre and geochemistry. So, it is important to study the exact explicit solutions of NEEs. It is also important to note that finding exact solutions of these equations is a difficult task and only in precise cases one can derive the solutions explicitly. However, an elephantine amount of work has been reported in the past few decades and significant progress has been made in acquiring exact explicit solutions of NEEs. In order to derive the exact explicit solutions, a collection of methods has been introduced by many researchers. Some of the popular methods encompass the solitary wave ansatz method, Jacobi elliptic function expansion method, the inverse scattering, Hirota's bilinear method, extended fan sub-equation method, homogeneous balance method, Lie group analysis, the unified method, etc. [1–11].

The one-dimensional Oskolkov equation is given by

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + u u_x = 0. \tag{1}$$

The one-dimensional modified Oskolkov equation is given by

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + u^2 u_x = 0. \tag{2}$$

Equations (1) and (2) are one-dimensional analogues of the following Oskolkov system:

$$(1 - \lambda\nabla^2)u_t = \alpha\nabla^2 u - (u \cdot \nabla)u - \nabla^2 p + f,\ \nabla \cdot u = 0. \tag{3}$$

System (3) governs the nonlinear dynamics of an incompressible viscoelastic Kelvin–Voigt fluid. It is important to note that the parameter λ can be negative and its negativeness is physically meaningful [12,13]. In this case, α is the coefficient of viscosity.

Recently, much attention has been paid to investigate NEEs in the presence of external periodic perturbation [14,15]. It is important to note that a completely integrable NEE cannot provide chaotic behaviour, but if we add an external periodic perturbation to an integrable NEE, then, it may show chaotic motions, for example, perturbed KdV-Burgers equation [16], perturbed sine-Gordon equation [17], perturbed Schrödinger equation [18]. Very recently, some researchers have reported bifurcation and chaotic behaviours of nonlinear waves in diverse fields of nonlinear science [19–24].

In the present work, we consider the generalised Oskolkov equation. Applying the unified method, we obtain the shock wave solutions. We discuss the conservation laws for the generalised Oskolkov equation. We employ numerical simulation to study the generalised Oskolkov equation. We mainly apply collocation finite element method to the generalised Oskolkov equation which is based on well-defined cubic B-spline functions. Furthermore, the accuracy of the proposed method is checked by two test problems including the motion of shock wave and evolution of waves with Gaussian and undular bore initial conditions. For this purpose, appropriate tabular and graphical results are recorded and presented. We also investigate the dynamic motions of the generalised Oskolkov equations in the presence of an external periodic perturbation. Using phase portraits and time series plots, we present the period-3, quasiperiodic and chaotic motions of the perturbed generalised Oskolov equation depending on the physical parameters.

The organisation of the paper is as follows: In §2, we describe the unified method. We consider the generalised Oskolkov equation and obtain the shock wave solution using the unified method in §3. In §4, we consider the conservation laws. In §5, we perform numerical simulation to check the accuracy of the proposed method. In §6, we investigate the dynamic motions of the generalised Oskolkov equations in the presence of an external periodic perturbation. Finally, we give a brief conclusion in §7.

2. The unified method

Gozukizil *et al* described the unified method for finding solutions of nonlinear partial differential equations (NPDEs) [25]. Suppose that a nonlinear partial differential equation (NPDE), say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0, \tag{4}$$

where $u\,(x, t)$ is an unknown function, P is a polynomial in $u = u\,(x, t)$ and its various partial derivatives, in which highest-order derivative and nonlinear terms are involved.

The summary of the unified method can be presented in the following six steps:

Step 1: To find the travelling wave solutions of eq. (4), using the wave variable

$$u(x, t) = U(\xi)\,, \quad \xi = x - ct, \tag{5}$$

where the constant c is the wave velocity. Substituting eq. (5) into eq. (4), the following ordinary differential equation (ODE) in ξ is obtained (which illustrates the principal advantage of a travelling wave solution, i.e., a partial differential equation (PDE) is reduced to an ordinary differential equation (ODE)).

$$P(U, cU', U', cU'', c^2U'', U'', \ldots) = 0. \tag{6}$$

Step 2: If necessary, one integrates eq. (6) as many times as possible and set the constants of integration to zero for simplicity.

Step 3: Suppose the solution of NPDE can be expressed by an ansatz as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{M} [a_i\phi^i + b_i\phi^{-i}], \tag{7}$$

where $\phi = \phi\,(\xi)$ satisfies the Riccati differential equation

$$\phi'(\xi) = \phi^2(\xi) + k, \tag{8}$$

where $\phi' = \mathrm{d}\phi/\mathrm{d}\xi$, and a_i, b_i and k are constants. The general solutions of eq. (8) are as follows:

Family 1. When $k < 0$, the solutions of eq. (8):

$$\phi\,(\xi) = \begin{cases} \dfrac{\sqrt{-(a^2+b^2)k} - a\sqrt{-k}\cosh\left(2\sqrt{-k}(\xi+\xi_0)\right)}{a\sinh\left(2\sqrt{-k}(\xi+\xi_0)\right)+b} \\ \dfrac{-\sqrt{-(a^2+b^2)k} - a\sqrt{-k}\cosh\left(2\sqrt{-k}(\xi+\xi_0)\right)}{a\sinh\left(2\sqrt{-k}(\xi+\xi_0)\right)+b} \\ \sqrt{-k} + \dfrac{-2a\sqrt{-k}}{a+\cosh\left(2\sqrt{-k}(\xi+\xi_0)\right)-\sinh\left(2\sqrt{-k}(\xi+\xi_0)\right)} \\ -\sqrt{-k} + \dfrac{2a\sqrt{-k}}{a+\cosh\left(2\sqrt{-k}(\xi+\xi_0)\right)+\sinh\left(2\sqrt{-k}(\xi+\xi_0)\right)} \end{cases} \tag{9}$$

where $a \neq 0$ and b are two real arbitrary constants, and ξ_0 an arbitrary constant.

Family 2. When $k > 0$, the solutions of eq. (8):

$$\phi(\xi) = \begin{cases} \dfrac{\sqrt{(a^2-b^2)k}-a\sqrt{k}\cos\left(2\sqrt{k}(\xi+\xi_0)\right)}{a\sin\left(2\sqrt{k}(\xi+\xi_0)\right)+b}, \\ \dfrac{-\sqrt{(a^2-b^2)k}-a\sqrt{k}\cos\left(2\sqrt{k}(\xi+\xi_0)\right)}{a\sin\left(2\sqrt{k}(\xi+\xi_0)\right)+b}, \\ i\sqrt{k}+\dfrac{-2ai\sqrt{k}}{a+\cos\left(2\sqrt{k}(\xi+\xi_0)\right)-i\sin\left(2\sqrt{k}(\xi+\xi_0)\right)}, \\ -i\sqrt{k}+\dfrac{2ai\sqrt{k}}{a+\cos\left(2\sqrt{k}(\xi+\xi_0)\right)+i\sin\left(2\sqrt{k}(\xi+\xi_0)\right)}, \end{cases} \tag{10}$$

where $a \neq 0$ and b are two real arbitrary constants, and ξ_0 an arbitrary constant.

Family 3. When $k = 0$, the solution of eq. (8):

$$\phi(\xi) = -\frac{1}{\xi+\xi_0}, \tag{11}$$

where ξ_0 is an arbitrary constant.

Gozukizil *et al* [25] have pointed out that the unified method gives the same solutions in Family 1 and Family 2. That is, when the relations $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$ are taken into account, the solutions in (9) and (10) are exactly the same. Therefore, the solutions in (9) and (10) can be converted to one another easily.

Step 4: The positive integer M can be accomplished by considering the homogeneous balance between the linear term of the highest order with the nonlinear term of highest degree appearing in eq. (6) as follows:

If the degree of $u(\xi)$ is defined as $D[u(\xi)] = M$, then the degree of other expressions is defined by

$$D\left[\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right] = M + q, \tag{12}$$

$$D\left[u^r\left(\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right)^s\right] = Mr + s(q+M). \tag{13}$$

Therefore, the value of M in eq. (7) can be found easily when using (12) and (13) in the obtained ODE in eq. (6).

Step 5: Substituting eqs (7) and (8) into eq. (6) and collecting all terms with the same degree of ϕ together, then setting each coefficient of terms with $\phi^i\,(-M \le i \le M)$ to zero yield a set of algebraic equations for $a_i,\ b_i,\ c$ and k.

Step 6: Substituting $a_i,\ b_i,\ c$ and k into eq. (7) which is obtained in Step 5 and using the general solutions of eq. (8) in eqs (9), (10) and (11), explicit solutions of eq. (4) can be obtained immediately depending on the value k.

3. The generalised Oskolkov equation

The generalised Oskolkov equation is given by

$$u_t + \alpha\left(u^p\right)_x + \beta u_{xx} + \gamma u_{xxt} = 0. \tag{14}$$

Using the wave variable $\xi = x - ct$ in eq. (14) and $U(\xi) = u(x,t)$, then integrating this equation and considering the integration constant to be zero, we obtain

$$-cU + \alpha U^p + \beta U' - \gamma c U'' = 0. \tag{15}$$

Balancing U^p and U'' gives $M = 2/(p-1)$. Using the transformation $U = V^{2/(p-1)}$ can be reached in a more simplified form of eq. (15) as follows:

$$U = V^{2/(p-1)}, \tag{16}$$

$$U' = \frac{2}{p-1}V^{((2/(p-1))-1)}V', \tag{17}$$

$$U'' = \frac{2}{p-1}\left(\frac{2}{p-1}-1\right)V^{((2/(p-1))-2)}\left(V'\right)^2 + \frac{2}{p-1}V^{((2/(p-1))-1)}V''. \tag{18}$$

Substituting the transformations (16)–(18) into eq. (15), we obtain

$$-cV^{2/(p-1)} + \alpha V^{2n/(p-1)} + \beta\frac{2}{p-1}V^{((2/(p-1))-1)}V' - \gamma c\left[\frac{2}{p-1}\left(\frac{2}{p-1}-1\right)V^{((2/(p-1))-2)}\left(V'\right)^2 + \frac{2}{p-1}V^{((2/(p-1))-1)}V''\right] = 0. \tag{19}$$

Multiplying eq. (19) with $V^{(2-(2/(p-1)))}$ results in

$$-cV^2 + \alpha V^4 + \beta\frac{2}{p-1}VV' - \gamma c\left[\frac{2}{p-1}\left(\frac{2}{p-1}-1\right)\left(V'\right)^2 + \frac{2}{p-1}VV''\right] = 0. \tag{20}$$

In this simplest form of the equation, we can balance V^4 and VV'' that gives $N = 1$. Therefore, the solutions of eq. (15) can be written in the form

$$U(\xi) = (V(\xi))^{2/(p-1)}, \tag{21}$$

$$V(\xi) = b_1\phi^{-1} + a_0 + a_1\phi, \tag{22}$$

where b_1, a_0, a_1 are constants which are to be determined later.

Substituting eq. (22) and its derivatives into eq. (20) and equating each coefficient of ϕ^i $(-1 \le i \le 1)$ to zero, we obtain a set of nonlinear algebraic equations for b_1, a_0, a_1 and c. Solving this system using Maple, we obtain

Set 1.

$$a_1 = \mp\frac{\gamma c\sqrt{\alpha c\left(\beta^2 - 4c^2\gamma\right)}}{\alpha\left(\beta^2 - 4c^2\gamma\right)},$$

$$a_0 = -\frac{\beta a_1(p-1)}{2c\gamma\,(p+3)},$$

$$c = \mp\frac{\beta\sqrt{2\gamma\,(p+1)}}{\gamma\,(p+3)},$$

$$k = -\frac{p^2 - 2p + 1}{8\gamma\,(p+1)},$$

$$b_1 = 0. \tag{23}$$

Set 2.

$$b_1 = \frac{\beta^2(p+1)}{8\alpha\gamma c a_1\left(p^2 + 6p + 9\right)},$$

$$a_1 = \mp\frac{\gamma c\sqrt{\alpha c\left(\beta^2 - 4c^2\gamma\right)}}{\alpha\left(\beta^2 - 4c^2\gamma\right)},$$

$$a_0 = -\frac{\beta a_1\,(p-1)}{2c\gamma\,(p+3)},$$

$$c = \mp\frac{\beta\sqrt{2\gamma\,(p+1)}}{\gamma(p+3)},$$

$$k = -\frac{p^2 - 2p + 1}{32\gamma(p+1)}. \tag{24}$$

Using these values and assuming $k \neq 0$, we obtain the following general solutions for Set 1:

$$u_1\,(x,t) = \left[A + BD\left(\frac{\mu}{2} - \frac{a\mu}{a + \cosh[\mu\,(x-ct)] - \sinh[\mu\,(x-ct)]}\right)\right]^{2/(p-1)}, \tag{25}$$

$$u_2\,(x,t) = \left[A + BD\left(-\frac{\mu}{2} + \frac{a\mu}{a + \cosh[\mu\,(x-ct)] + \sinh[\mu\,(x-ct)]}\right)\right]^{2/(p-1)}, \tag{26}$$

$$u_3\,(x,t) = \left[A + BD\frac{\left((\mu/2)\sqrt{\left(a^2+b^2\right)} - (a\mu/2)\cosh[\mu\,(x-ct)]\right)}{(b + a\sinh[\mu\,(x-ct)])}\right]^{2/(p-1)}, \tag{27}$$

$$u_4\,(x,t) = \left[A + BD\frac{\left(-(\mu/2)\sqrt{\left(a^2+b^2\right)} - (a\mu/2)\cosh[\mu\,(x-ct)]\right)}{(b + a\sinh[\mu\,(x-ct)])}\right]^{2/(p-1)}, \tag{28}$$

where $k < 0$, and

$$A = -\frac{1}{2}\frac{\beta\,(p-1)\sqrt{\frac{\alpha\beta\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(p+3)}}}{\alpha\,(p+3)\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)},$$

$$B = \frac{1}{\alpha\,(p+3)\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)},$$

$$D = \beta\sqrt{2\gamma\,(p+1)}$$

$$\times\sqrt{\frac{\alpha\beta\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma\,(p+1)}}{\gamma\,(p+3)}},$$

$$\mu = \frac{\sqrt{8}}{4}\sqrt{\frac{p^2 - 2p + 1}{\gamma\,(p+1)}},$$

$$c = \mp\frac{\beta\sqrt{2\gamma\,(p+1)}}{\gamma\,(p+3)}, \tag{29}$$

and $a \neq 0$ and b are two real arbitrary constants.

Likewise, using these values and assuming $k \neq 0$, we obtain following general solutions for Set 2:

$$u_5(x,t) = \left[\frac{E}{F\left(\frac{\mu}{2} - \frac{a\mu}{a+\cosh[\mu(x-ct)]-\sinh[\mu(x-ct)]}\right)} + A \right.$$
$$\left. + BD\left(\frac{\mu}{2} - \frac{a\mu}{a+\cosh[\mu(x-ct)]-\sinh[\mu(x-ct)]}\right)\right]^{2/(p-1)}, \tag{30}$$

$$u_6(x,t) = \left[\frac{E}{F\left(-\frac{\mu}{2} + \frac{a\mu}{a+\cosh[\mu(x-ct)]+\sinh[\mu(x-ct)]}\right)} + A \right.$$
$$\left. + BD\left(-\frac{\mu}{2} + \frac{a\mu}{a+\cosh[\mu(x-ct)]+\sinh[\mu(x-ct)]}\right)\right]^{2/(p-1)}, \tag{31}$$

$$u_7(x,t) = \left[\frac{E(b+a\sinh[\mu(x-ct)])}{F((\mu/2)\sqrt{(a^2+b^2)} - (a\mu/2)\cosh[\mu(x-ct)])} + A \right.$$
$$\left. + BD\frac{((\mu/2)\sqrt{(a^2+b^2)} - (a\mu/2)\cosh[\mu(x-ct)])}{(b+a\sinh[\mu(x-ct)])}\right]^{2/(p-1)}, \tag{32}$$

$$u_8(x,t) = \left[\frac{E(b+a\sinh[\mu(x-ct)])}{F(-(\mu/2)\sqrt{(a^2+b^2)} - (a\mu/2)\cosh[\mu(x-ct)])} + A \right.$$
$$\left. + BD\frac{(-(\mu/2)\sqrt{(a^2+b^2)} - (a\mu/2)\cosh[\mu(x-ct)])}{(b+a\sinh[\mu(x-ct)])}\right]^{2/(p-1)}, \tag{33}$$

where $k < 0$, and

$$A = -\frac{1}{2}\frac{\beta(p-1)\sqrt{\frac{\alpha\beta\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(n+3)}}}{\alpha(p+3)\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right)},$$
$$B = \frac{1}{\alpha(p+3)\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right)},$$
$$D = \beta\sqrt{2\gamma(p+1)}\sqrt{\frac{\alpha\beta\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(p+3)}},$$
$$E = \frac{(p+3)^2}{16}\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right),$$
$$F = \gamma\sqrt{\frac{\alpha\beta\left(\beta^2-\frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(p+3)}}(p^2+6p+9),$$
$$\mu = \frac{\sqrt{32}}{16}\sqrt{\frac{p^2-2p+1}{\gamma(p+1)}},$$
$$c = \mp\frac{\beta\sqrt{2\gamma(p+1)}}{\gamma(p+3)}, \tag{34}$$

and $a \neq 0$ and b are two real arbitrary constants. Specifically, for $p = 2$ in eq. (14), the solutions are

$$u_{1.1}(x,t) = \left[A + B\left(\frac{\mu}{2} - \frac{a\mu}{a + \cosh[\mu(x-ct)] - \sinh[\mu(x-ct)]}\right)\right]^2, \tag{35}$$

$$u_{2.1}(x,t) = \left[A + B\left(-\frac{\mu}{2} + \frac{a\mu}{a + \cosh[\mu(x-ct)] + \sinh[\mu(x-ct)]}\right)\right]^2, \tag{36}$$

$$u_{3.1}(x,t) = \left[A + \frac{B\left((\mu/2)\sqrt{(a^2+b^2)} - a(\mu/2)\cosh[\mu(x-ct)]\right)}{(b + a\sinh[\mu(x-ct)])}\right]^2, \tag{37}$$

$$u_{4.1}(x,t) = \left[A + \frac{B\left(-(\mu/2)\sqrt{(a^2+b^2)} - a(\mu/2)\cosh[\mu(x-ct)]\right)}{(b + a\sinh[\mu(x-ct)])}\right]^2, \tag{38}$$

where $k < 0$, and

$$A = -\frac{\sqrt{125\sqrt{6}\alpha\beta^3/\sqrt{\gamma}}}{50\alpha\beta},$$

$$B = \frac{\sqrt{6\gamma}\sqrt{125\sqrt{6}\alpha\beta^3/\sqrt{\gamma}}}{25\alpha\beta},$$

$$\mu = \frac{1}{12}\sqrt{\frac{24}{\gamma}},$$

$$c = \mp\frac{\sqrt{6}\beta}{5\sqrt{\gamma}}, \tag{39}$$

and $a \neq 0$ and b are two real arbitrary constants.

$$u_{5.1}(x,t) = \left[\frac{D}{\left(\frac{\mu}{2} - \frac{a\mu}{a+\cosh[\mu(x-ct)]-\sinh[\mu(x-ct)]}\right)} + A + B\left(\frac{\mu}{2} - \frac{a\mu}{a + \cosh[\mu(x-ct)] - \sinh[\mu(x-ct)]}\right)\right]^2, \tag{40}$$

$$\begin{aligned} u_{6.1}(x,t) = &\left[\frac{D}{\left(-\frac{\mu}{2} + \frac{a\mu}{a+\cosh[\mu(x-ct)]+\sinh[\mu(x-ct)]}\right)} + A\right. \\ &\left. + B\left(-\frac{\mu}{2} + \frac{a\mu}{a + \cosh[\mu(x-ct)] + \sinh[\mu(x-ct)]}\right)\right]^2, \end{aligned} \tag{41}$$

$$\begin{aligned} u_{7.1}(x,t) = &\left[\frac{D(b + a\sinh[\mu(x-ct)])}{((\mu/2)\sqrt{a^2+b^2} - (a\mu/2)\cosh[\mu(x-ct)])}\right. \\ &\left. + A + \frac{B((\mu/2)\sqrt{a^2+b^2} - (a\mu/2)\cosh[\mu(x-ct)])}{(b + a\sinh[\mu(x-ct)])}\right]^2, \end{aligned} \tag{42}$$

$$\begin{aligned} u_{8.1}(x,t) = &\left[\frac{D(b + a\sinh[\mu(x-ct)])}{(-(\mu/2)\sqrt{a^2+b^2} - (a\mu/2)\cosh[\mu(x-ct)])}\right. \\ &\left. + A + \frac{B(-(\mu/2)\sqrt{a^2+b^2} - (a\mu/2)\cosh[\mu(x-ct)])}{(b + a\sinh[\mu(x-ct)])}\right]^2, \end{aligned} \tag{43}$$

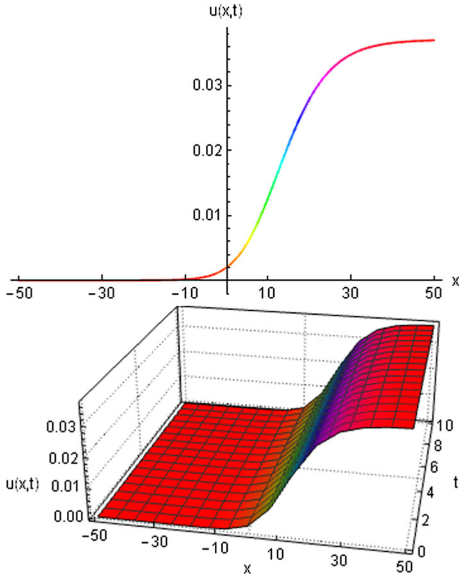

Figure 1. Shock wave profile for $p = 2, \alpha = 0.5, \beta = 0.1, \gamma = 7$ and $a = 0.3$.

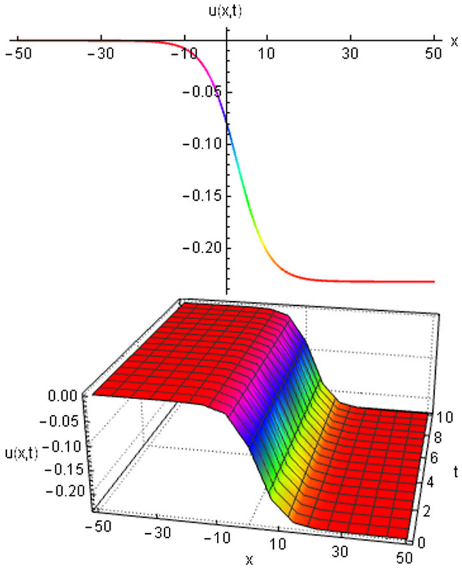

Figure 2. Shock wave profile for $p = 3, \alpha = 0.33, \beta = 0.1, \gamma = 7$ and $a = 0.5$.

where $k < 0$, and

$$A = -\frac{\sqrt{125\sqrt{6}\alpha\beta^3/\sqrt{\gamma}}}{50\alpha\beta},$$

$$B = \frac{\sqrt{6\gamma}\sqrt{125\sqrt{6}\alpha\beta^3/\sqrt{\gamma}}}{25\alpha\beta},$$

$$D = \frac{\sqrt{125\beta^2}}{400\gamma\sqrt{\alpha\beta^3\sqrt{6}/\sqrt{\gamma}}},$$

$$\mu = \frac{1}{48}\sqrt{\frac{96}{\gamma}},$$

$$c = \mp\frac{\sqrt{6}\beta}{5\sqrt{\gamma}}, \tag{44}$$

and $a \neq 0$ and b are two real arbitrary constants. Shock wave solutions of the generalised Oskolkov equation for different values of p are shown in figures 1 and 2.

4. Conservation laws

For a conserved flow (T^x, T^t) for which $D_x T^x + D_t T^t = 0$ on the solutions of eq. (14), we employ the multiplier approach in which

$$\mathcal{E}[Q(x,t,u,u_x,u_t)(u_t + \alpha(u^p)_x + \beta u_{xx} + \gamma u_{xxt})], \tag{45}$$

vanishes, where $\mathcal{E}$ is the Euler operator. Each multiplier leads to a conserved flow.

There is just one multiplier that leads to non-trivial conservation laws, viz., $Q = 1$, with conserved vector components

$$T^t = u, \tag{46}$$

$$T^x = \alpha u^p + \beta u_x + \gamma u_{xt}. \tag{47}$$

Equation (14) admits two point symmetry generators ∂_t and ∂_x.

There are, at least, four multipliers that lead to non-trivial conservation laws for the case $p = 1$, viz.,

$$Q_1 = 1, \tag{48}$$

$$Q_2 = \mathrm{e}^{\frac{\alpha}{\beta}x}, \tag{49}$$

$$Q_3 = \mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma - x\alpha - x\sqrt{\alpha^2+4\beta-4\gamma} - 2t\beta}{-\beta+\gamma}\right)}, \tag{50}$$

$$Q_4 = \mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma - x\alpha + x\sqrt{\alpha^2+4\beta-4\gamma} - 2t\beta}{-\beta+\gamma}\right)}. \tag{51}$$

The corresponding conserved flows for the first three cases are, respectively,

$$T_1^t = u, \tag{52}$$

$$T_1^x = \alpha u + \beta u_x + \gamma u_{xt}, \tag{53}$$

$$T_2^t = -\frac{1}{3}\mathrm{e}^{x\alpha/\beta}(u_{1,1}\gamma\beta^2 - u_1\gamma\alpha\beta + u\alpha^2\gamma + 3u\beta^2)\beta^{-2}, \tag{54}$$

$$T_2^x = \frac{1}{3}\,\mathrm{e}^{x\alpha/\beta}(-2\,u_{1,2}\gamma\beta + u_2\gamma\alpha - 3\,u_1\beta^2)\beta^{-1}, \tag{55}$$

$$\begin{aligned} T_3^t = {} & -\frac{1}{6(\gamma-\beta)^2}\mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma - x\alpha - x\sqrt{\alpha^2+4\beta-4\gamma}-2t\beta}{-\beta+\gamma}\right)} \\ & \times \left[2\,u_{1,1}\gamma\beta^2 - 4\,u_{1,1}\gamma^2\beta + 2\,u_{1,1}\gamma^3 - u_1\gamma\alpha\beta\right. \\ & + u_1\gamma^2\alpha - u_1\gamma\sqrt{\alpha^2+4\,\beta-4\gamma\beta} \\ & + u_1\gamma^2\sqrt{\alpha^2+4\,\beta-4\,\gamma} + u\alpha^2\gamma \\ & + u\alpha\sqrt{\alpha^2+4\,\beta-4\,\gamma\gamma} - 10\,u\beta\gamma \\ & \left. + 4\,u\gamma^2 + 6\,u\beta^2\right], \end{aligned} \tag{56}$$

$$\begin{aligned} T_3^x = {} & \frac{1}{6(\gamma-\beta)}\mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma - x\alpha - x\sqrt{\alpha^2+4\beta-4\gamma}-2t\beta}{-\beta+\gamma}\right)}[4\,u_{1,2}\gamma\beta \\ & - 4\,u_{1,2}\gamma^2 - u_2\gamma\alpha \\ & - u_2\gamma\sqrt{\alpha^2+4\,\beta-4\,\gamma} - 8\,u_1\gamma\beta \\ & + 2\,u_1\gamma^2 + 6\,u_1\beta^2 - 4\,u\alpha\gamma \\ & + 2\,u\sqrt{\alpha^2+4\,\beta-4\,\gamma\gamma} + 3\,u\alpha\beta \\ & - 3\,u\sqrt{\alpha^2+4\,\beta-4\,\gamma\beta}]. \end{aligned} \tag{57}$$

5.1 *Governing equation and cubic B-spline basis functions*

We consider the generalised Oskolkov equation (14) with the physical boundary conditions $u \to 0$ as $x \to \pm\infty$, where α, β and γ are arbitrary parameters and the subscripts x and t denote the differentiation.

To implement the numerical method, solution domain of the problem is restricted over an interval $a \le x \le b$. Firstly, we subdivide the interval $[a, b]$ into N intervals with the length h, by introducing the knots of subdivision $x_0, x_1, \ldots, x_N$, such that $a = x_0 < x_1 < \cdots < x_N = b$. Lengths of these finite elements are $h = (b-a)/N = (x_{m+1} - x_m)$ for $m = 1, 2, \ldots, N$. Boundary conditions have been selected from the following homogeneous boundary conditions:

$$u_N(a,t) = 0, \quad u_N(b,t) = 0, \quad t > 0 \tag{58}$$

and the initial condition

$$u(x,0) = f(x), \quad a \le x \le b. \tag{59}$$

The cubic B-splines $\phi_m(x)$, $(m = -1(1)N+1)$, at the knots x_m are defined over the interval $[a, b]$ by the relations [26]:

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0, & \text{elsewhere}. \end{cases} \tag{60}$$

The case $p = 1$ admits the following symmetry generators, ∂_t, ∂_x, $u\partial_u$, $\mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma+2t\beta-x\alpha+x\sqrt{\alpha^2+4\beta-4\gamma}}{\beta+\gamma}\right)}\partial_u$ and $\mathrm{e}^{\frac{1}{2}\left(\frac{2t\gamma+2t\beta-x\alpha-x\sqrt{\alpha^2+4\beta-4\gamma}}{\beta+\gamma}\right)}\partial_u$.

5. Numerical simulations

In this section, we have done the numerical study of generalised Oskolkov equation (14). Here, we apply collocation finite element method to the generalised Oskolkov equation (14) which is based on the well-defined cubic B-spline functions. The accuracy of the proposed method is checked by two test problems including the motion of shock wave and evolution of waves with Gaussian and undular bore initial conditions. Appropriate tabular and graphical results are recorded and presented in the proceeding sections.

The set of functions $\{\phi_{-1}(x), \phi_0(x), \ldots, \phi_N(x), \phi_{N+1}(x)\}$ forms a basis for functions defined over $[a, b]$. The approximate solution $u_N(x,t)$ to the exact solution $u(x,t)$ is given by

$$u_N(x,t) = \sum_{i=-1}^{N+1} \phi_i(x)\delta_i(t), \tag{61}$$

where $\delta_i(t)$ are time-dependent parameters to be determined from the boundary and collocation conditions. Each cubic B-spline covers four elements, so that each element $[x_m, x_{m+1}]$ is covered by four splines. The values of $\phi_m(x)$ and its derivatives are tabulated in table 1.

Using trial function (61) and cubic B-splines (60), the values of u, u' and u'' at the knots are determined in terms of the element parameters δ_m by

$$u_m = u(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1},$$

$$u'_m = u'(x_m) = \frac{3}{h}(-\delta_{m-1} + \delta_{m+1}),$$

Table 1. Cubic B-spline function and its derivatives at nodes x_m.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$\phi_m(x)$	0	1	4	1	0
$h\phi'_m(x)$	0	-3	0	3	0
$h^2\phi''_m(x)$	0	6	-12	6	0

$$u''_m = u''(x_m) = \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}), \tag{62}$$

where the symbols $'$ and $''$ denote first and second differentiation with respect to x, respectively. The splines $\phi_m(x)$ and its four principle derivatives vanish outside the interval $[x_{m-2}, x_{m+2}]$.

5.2 *Collocation finite element method*

Substituting the nodal values of u_m and its spatial derivatives given by eq. (62) into eq. (14), the following general form of the solution approach is obtained for the linearisation technique:

$$\begin{aligned}&\dot{\delta}_{m-1} + 4\dot{\delta}_m + \dot{\delta}_{m+1} + \frac{3}{h}\alpha Z_m(-\delta_{m-1} + \delta_{m+1})\\ &\quad + \frac{6}{h^2}\beta(\delta_{m-1} - 2\delta_m + \delta_{m+1})\\ &\quad + \frac{6}{h^2}\gamma(\dot{\delta}_{m-1} - 2\dot{\delta}_m + \dot{\delta}_{m+1}) = 0,\end{aligned} \tag{63}$$

where

$$Z_m = pu_m^{p-1} = p\,(\delta_{m-1} + 4\delta_m + \delta_{m+1})^{p-1}, \tag{64}$$

and $\cdot$ indicates derivative with respect to t. For the linearisation technique, the term pu^{p-1} in non-linear term $pu^{p-1}u_x$ is taken as eq. (63) by assuming that the quantity pu^{p-1} is locally constant.

If time parameters δ_i's and its time derivatives $\dot{\delta}_i$'s in eq. (63) are discretised by the Crank–Nicolson formula and usual finite difference approximation, respectively:

$$\delta_i = \frac{\delta_i^{n+1} + \delta_i^n}{2}, \quad \dot{\delta}_i = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}. \tag{65}$$

We obtain a recurrence relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-1, \ldots, m+1$

$$\begin{aligned}&\sigma_1\delta_{m-1}^{n+1} + \sigma_2\delta_m^{n+1} + \sigma_3\delta_{m+1}^{n+1}\\ &\quad = \sigma_4\delta_{m-1}^n + \sigma_5\delta_m^n + \sigma_6\delta_{m+1}^n,\end{aligned} \tag{66}$$

where

$$\begin{aligned}&\sigma_1 = [1 - EZ_m + M + K],\\ &\sigma_2 = [4 - M - K],\\ &\sigma_3 = [1 + EZ_m + M + K],\\ &\sigma_4 = [1 + EZ_m - M + K],\\ &\sigma_5 = [4 + M - K],\\ &\sigma_6 = [1 - EZ_m - M + K],\\ &m = 0, 1, \ldots, N,\\ &E = \frac{3}{2h}\alpha\Delta t, \quad M = \frac{6}{h^2}\beta\Delta t, \quad M = \frac{6}{h^2}\gamma.\end{aligned} \tag{67}$$

System (66) consists of $(N+1)$ linear equations including $(N+3)$ unknown parameters $(\delta_{-1}, \delta_0, \ldots, \delta_N, \delta_{N+1})^T$. To obtain a unique solution to this system, we need two additional constraints. These are obtained from the boundary conditions and can be used to eliminate δ_{-1} and δ_{N+1} from system (66) which later becomes a matrix equation for the $N+1$ unknowns $d = (\delta_0, \delta_1, \ldots, \delta_N)^T$ of the form

$$A\mathbf{d}^{\mathbf{n+1}} = B\mathbf{d}^{\mathbf{n}}. \tag{68}$$

The matrices A and B are pentagonal $(N+1) \times (N+1)$ matrices and this matrix can be solved by using the pentagonal algorithm. However, two or three inner iterations are implemented to the term $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to cope with the non-linearity caused by Z_m. Before the solution process begins iteratively, the initial vector $\mathbf{d}^{\mathbf{0}} = (\delta_0, \delta_1, \ldots, \delta_{N-1}, \delta_N)$ must be determined by using the initial condition and the following derivative at the boundary condition:

$$\begin{aligned}&u_N(x, 0) = u\,(x_m, 0),\ m = 0, 1, 2, \ldots, N\\ &(u_N)_x\,(a, 0) = 0, \qquad (u_N)_x\,(b, 0) = 0.\end{aligned} \tag{69}$$

So we have the following matrix form of the initial vector $\mathbf{d}^{\mathbf{0}}$:

$$W\mathbf{d}^{\mathbf{0}} = C, \tag{70}$$

where

$$\begin{bmatrix} 4 & 2 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 & 1 \\ & & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{N-2} \\ \delta_{N-1} \\ \delta_N \end{bmatrix} = \begin{bmatrix} U\,(x_0, 0) \\ U\,(x_1, 0) \\ U\,(x_2, 0) \\ \vdots \\ U\,(x_{N-2}, 0) \\ U\,(x_{N-1}, 0) \\ U\,(x_N, 0) \end{bmatrix}.$$

This matrix system can be solved efficiently by using a variant of Thomas algorithm.

5.3 *Stability analysis*

The stability analysis is based on the von Neumann theory. The growth factor ξ of the error in a typical mode of amplitude

$$\delta_m^n = \xi^n e^{imkh}, \tag{71}$$

is determined from a linearisation of the numerical scheme where k is the mode number and h the element size. Substituting the Fourier mode (71) into (66) gives the following equality:

$$\sigma_1 \xi^{n+1} e^{i(m-1)kh} + \sigma_2 \xi^{n+1} e^{i(m)kh} + \sigma_3 \xi^{n+1} e^{i(m+1)kh} = \sigma_4 \xi^n e^{i(m-1)kh} + \sigma_5 \xi^n e^{i(m)kh} + \sigma_6 \xi^n e^{i(m+1)kh}. \tag{72}$$

Now, when Euler's formula

$$e^{ikh} = \cos(kh) + i\sin(kh), \tag{73}$$

is implemented to simplify eq. (72), we get the following growth factor:

$$\xi = \frac{\omega_1 - i\varpi}{\omega_2 + i\varpi}, \tag{74}$$

in which

$$\begin{aligned} \omega_1 &= [(2 - 2M + 2K)\cos(kh) + (4 + M - K)], \\ \omega_2 &= [(2 + 2M + 2K)\cos(kh) + (4 - M - K)], \\ \varpi &= [(2EZ_m)\sin(kh)]. \end{aligned} \tag{75}$$

As $|\xi| \le 1$, the linearised scheme is unconditionally stable.

5.4 *Test problems*

Numerical results of the generalised Oskolkov equation are obtained for three problems: the motion of shock wave, evolution of waves with Gaussian and undular bore initial conditions. We use the error norm L_2 that is defined as

$$L_2 = \|u^{\text{exact}} - u_N\|_2 \simeq \sqrt{h \sum_{j=1}^{N} |u_j^{\text{exact}} - (u_N)_j|^2}, \tag{76}$$

and the error norm L_∞

$$L_\infty = \|u^{\text{exact}} - u_N\|_\infty \simeq \max_j |u_j^{\text{exact}} - (u_N)_j|, \quad j = 1, 2, \ldots, N, \tag{77}$$

to calculate the difference between analytical and numerical solutions at some specified times. The generalised Oskolkov equation (14) possesses only one invariant:

$$I = \int_a^b u \, dx \simeq h \sum_{j=1}^{N} u_j^n, \tag{78}$$

which corresponds to conservation laws. In the simulation of shock wave motion, the invariant I is monitored to check the conversation of the numerical algorithm.

5.4.1 *The motion of shock wave.* The shock wave solution of the generalised Oskolkov equation (14) is given by

$$u(x,t) = \left[A + BD\left(\frac{\mu}{2} - \frac{a\mu}{a + \cosh[\mu(x - ct)] - \sinh[\mu(x - ct)]}\right)\right]^{2/(p-1)}, \tag{79}$$

where

$$\begin{aligned} A &= -\frac{1}{2}\,\frac{\beta(p-1)\sqrt{\dfrac{\alpha\beta\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(p+3)}}}{\alpha(p+3)\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)}, \\ B &= \frac{1}{\alpha(p+3)\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)}, \\ D &= \beta\sqrt{2\gamma(p+1)}\sqrt{\frac{\alpha\beta\left(\beta^2 - \frac{8\beta^2(p+1)}{(p+3)^2}\right)\sqrt{2\gamma(p+1)}}{\gamma(p+3)}}, \\ \mu &= \frac{\sqrt{8}}{4}\sqrt{\frac{p^2 - 2p + 1}{\gamma(p+1)}}, \\ c &= \mp\frac{\beta\sqrt{2\gamma(p+1)}}{\gamma(p+3)}. \end{aligned} \tag{80}$$

Note that a, c, μ and p are arbitrary constants. The initial condition is

$$u(x,0) = \left[A + BD\left(\frac{\mu}{2} - \frac{a\mu}{a + \cosh(\mu x) - \sinh(\mu x)}\right)\right]^{2/(p-1)}. \tag{81}$$

To show the motion of the shock wave solution numerically, let $\beta = 0.1$, $\gamma = 7$, $h = \Delta = 0.1$ and different values of α, a and p over the interval $[-50, 50]$. We consider the following two cases:

Case I: For $p = 2$, the parameters chosen are: $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$, $a = 0.3$, $h = 0.1$ and $\Delta t = 0.1$. Also,

Table 2. Invariant and error norms for shock wave with $p = 2, \alpha = 0.5, \beta = 0.1, \gamma = 7, a = 0.3, h = \Delta t = 0.1$ and $-50 \le x \le 50$.

t	I	L_2-Error	L_∞-Error
0.0	1.3252394649	0.0000000000	0.0000000000
1.0	1.3064677652	0.0028324389	0.0005233176
2.0	1.2879619833	0.0056238627	0.0010392063
3.0	1.2697183517	0.0083748519	0.0015477715
4.0	1.2517331566	0.0110859790	0.0020491170
5.0	1.2517331566	0.0137578081	0.0025433452

Table 3. Invariant and error norms for shock wave with $p = 3, \alpha = 0.33, \beta = 0.1, \gamma = 7, a = 0.5, h = \Delta t = 0.1$ and $-50 \le x \le 50$.

t	I	L_2-Error	L_∞-Error
0.0	-10.9717745688	0.0000000000	0.0000000000
1.0	-10.8163641228	0.0213556686	0.0032787672
2.0	-10.6631552407	0.0424043554	0.0065110230
3.0	-10.5121167286	0.0631504075	0.0096974273
4.0	-10.3632178343	0.0835981104	0.0128386305
5.0	-10.2164282412	0.1037516888	0.0159353022

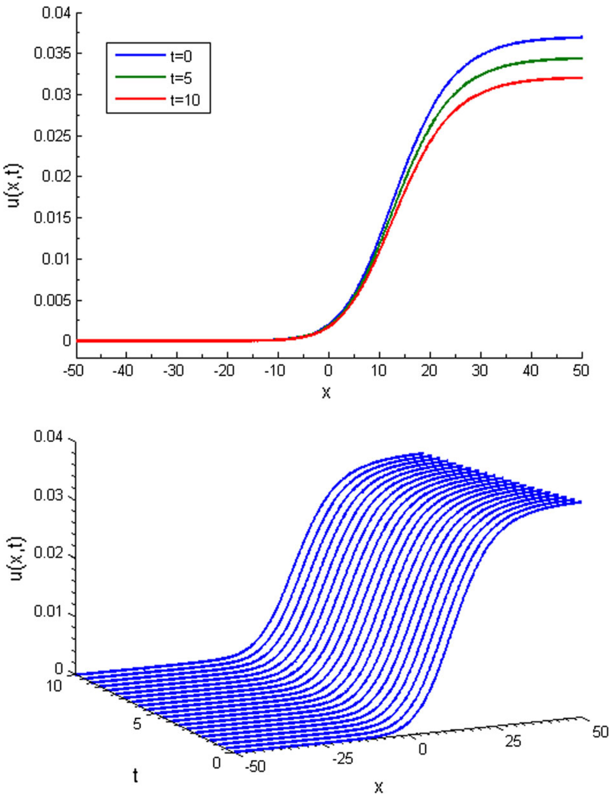

Figure 3. Shock wave profile for $p = 2$, $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$, $a = 0.3$ and $h = \Delta t = 0.1$.

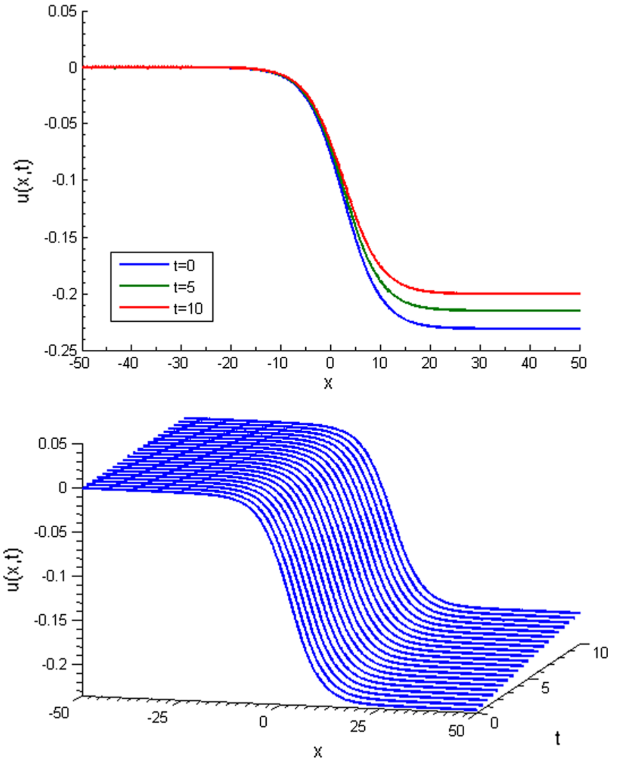

Figure 4. Shock wave profile for $p = 3$, $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7$, $a = 0.5$ and $h = \Delta t = 0.1$.

boundary condition is taken by $u \to 0$ as $x \to -\infty$ and $u \to 0.33$ as $x \to \infty$. The conserved quantity and error norms L_2 and L_∞ are shown at selected times up to time $t = 5$. The obtained results are tabulated in table 2. It can be seen from table 2 that the error norms L_2 and L_∞ are found to be small enough and the invariants are nearly unchanged as the time increases. The motion of shock wave profile is plotted from $t = 0$ to $t = 10$ in figure 3.

Case II: For $p = 3$, the parameters $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7, a = 0.5, h = 0.1$ and $\Delta t = 0.1$ are considered to examine the quantity of the invariant and error norms. Also, boundary condition is taken by $u \to -0.23$ as $x \to -\infty$ and $u \to 0$ as $x \to \infty$. The calculated values are presented in table 3. As can be seen in table 3, the error norms L_2 and L_∞ are reasonably small and the invariant remains almost constant as the time increases. The motion of shock wave is depicted from $t = 0$ to $t = 10$ in figure 4.

Finally, error distributions at specified times for $p = 2$ and $p = 3$ are depicted in figure 5 and the errors are shown between the analytical and numerical results over the problem domain.

In the light of all these studies, it is observed that as p increases, the error norms increase.

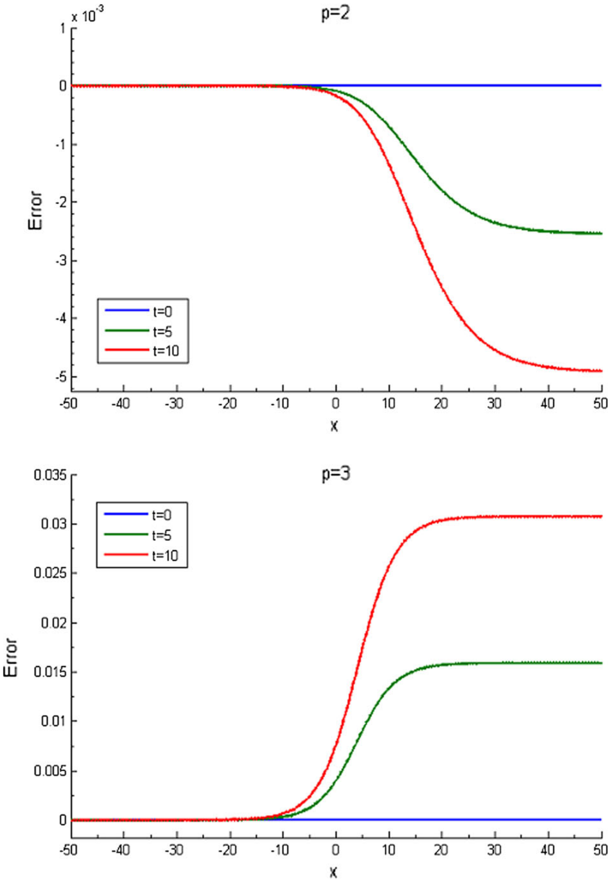

Figure 5. Errors for $p = 2$ and $p = 3$ with parameters in Cases I and II.

5.4.2 *Evolution of waves.* We observe the evolution of waves for the generalised Oskolkov equation (14) by using the Gaussian and undular bore initial conditions.

Gaussian initial condition
Evolution of waves for the generalised Oskolkov equation has been studied using the Gaussian initial condition

$$u(x, 0) = \exp(-x^2), \tag{82}$$

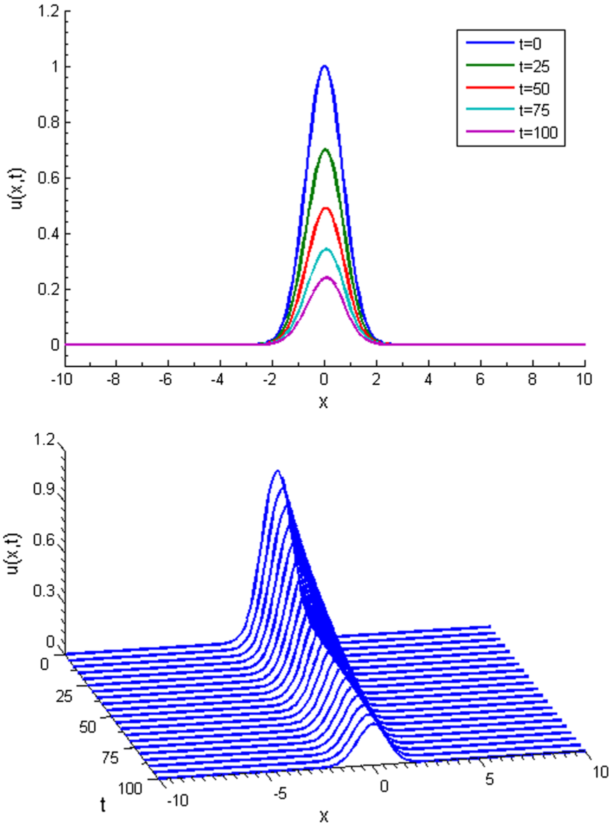

Figure 6. Generated wave profiles for $p = 2$, $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$, $a = 0.3$ and $h = \Delta t = 0.1$.

and the boundary condition

$$u(-10, t) = u(10, t) = 0, \quad t > 0 \tag{83}$$

for various values of h and Δt.

To show the evolution of waves numerically, let $\beta = 0.1$ and $\gamma = 7$ with different values of α, a, p, h and Δt over the interval $[-10, 10]$. We consider the following two cases:

Case I: For $p = 2$, parameters are taken as $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$ and $a = 0.3$ with various values of h and Δt.

Table 4. Invariants for Gaussian initial condition with $p = 2$, $p = 3$ and other parameters in Cases I and II.

t	$p = 2$		$p = 3$	
	$h = \Delta t = 0.1$ I	$h = \Delta t = 0.01$ I	$h = \Delta t = 0.1$ I	$h = \Delta t = 0.01$ I
0.0	1.7724537283	1.7724549574	1.7724537283	1.7724549574
1.0	1.7473485479	1.7473145334	1.7473481951	1.7473145334
2.0	1.7225989171	1.7225307001	1.7225975675	1.7225307001
3.0	1.6981998039	1.6980983996	1.6981968994	1.6980983996
4.0	1.6741462478	1.6740126458	1.6741413093	1.6740126458
5.0	1.6504333585	1.6502685232	1.6504259785	1.6502685232

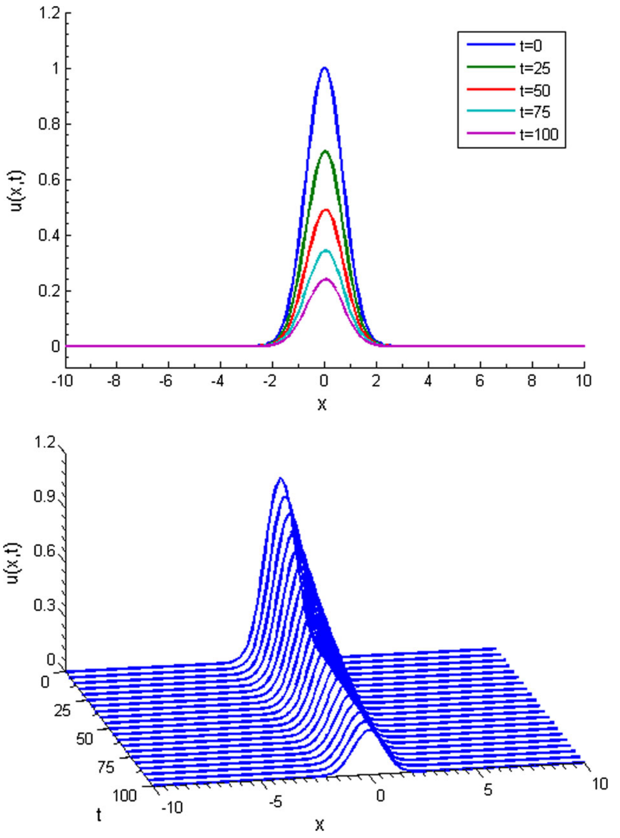

Figure 7. Generated waves profile for $p = 3$, $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7$, $a = 0.5$ and $h = \Delta t = 0.1$.

Numerical computations are calculated up to $t = 5$. The values of the conserved quantity of motion are presented for space and time steps in table 4. Figure 6 illustrates the development of the Gaussian initial condition into waves from $t = 0$ to $t = 100$.

Case II: For $p = 3$, parameters are taken as $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7$ and $a = 0.5$ with various values of h and Δt. Computations are carried out from $t = 0$ to $t = 5$. In table 4, the values of the conserved quantity of motion are tabulated for different values of space and time steps. Also, the development of the Gaussian initial condition into waves is showed from $t = 0$ to $t = 100$ in figure 7.

Consequently, as the time progresses, amplitudes of the waves decrease.

Undular bore initial condition

As the last test problem, evolution of waves for the generalised Oskolkov equation has been worked using the undular bore initial condition

$$u(x,0) = \frac{1}{2}U_0\left[1 - \tanh\left(\frac{|x| - x_0}{d}\right)\right] \tag{84}$$

and the boundary condition

$$u(-60,t) = u(60,t) = 0, \quad t > 0 \tag{85}$$

to produce waves for generalised Oskolkov equation. The undular bore reflects the elevation of water above the equilibrium surface at time $t = 0$. The change in water level of magnitude equation (14) is centred on $x = x_0$ and d measures the steepness of the change. The smaller the value of d, the steeper is the slope.

To show evolution of waves numerically, let $\beta = 0.1$, $\gamma = 7$, $U_0 = 0.25$, $x_0 = 25$ and $d = 5$ with different values of α, a, p, h and Δt over the interval $[-60, 60]$. We consider the following two cases:

Case I: For $p = 2$, parameters are taken as $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$ and $a = 0.3$ with various values of h and Δt. The program runs to time $t = 5$. The computed conserved quantity is presented in table 5. Figure 8 shows the development of waves from $t = 0$ to $t = 100$.

Case II: For $p = 3$, parameters are taken as $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7$ and $a = 0.5$ with various values of h and Δt. Simulation is calculated up to time $t = 5$. In table 5, the values of the conserved quantity of the motion are reported. The development of the waves is illustrated in figure 9 from $t = 0$ to $t = 100$.

Table 5. Invariants for undular bore initial condition with $p = 2$, $p = 3$ and other parameters in Cases I and II.

t	$p = 2$		$p = 3$	
	$h = \Delta t = 0.1$ I	$h = \Delta t = 0.02$ I	$h = \Delta t = 0.1$ I	$h = \Delta t = 0.02$ I
0.0	12.5000275676	12.4998649440	12.5000275676	12.4998649440
1.0	12.3229763044	12.3225749466	12.3229763042	12.3225749466
2.0	12.1484328081	12.1477995158	12.1484328073	12.1477995157
3.0	11.9763615584	11.9755029866	11.9763615566	11.9755029865
4.0	11.8067275384	11.8056501999	11.8067275352	11.8056501998
5.0	11.6394962269	11.6382064953	11.6394962219	11.6382064951

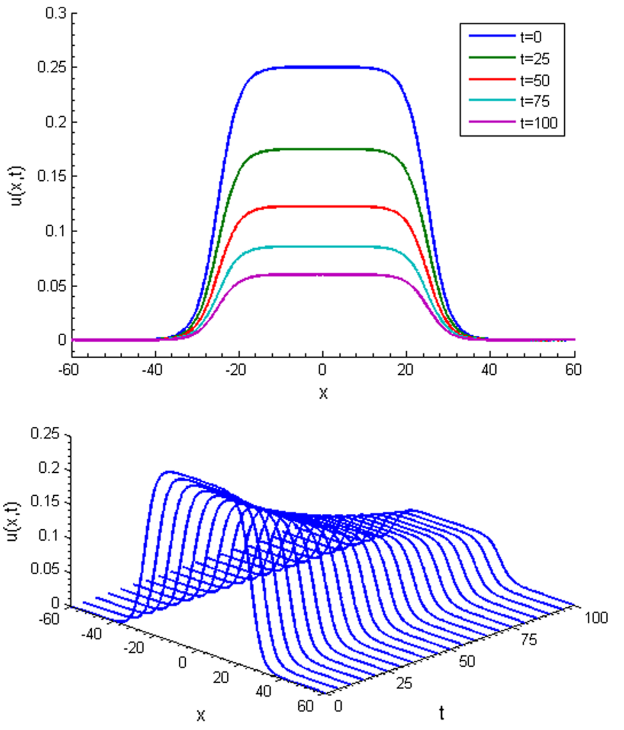

Figure 8. Generated waves profile for $p = 2$, $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 7$, $a = 0.3$ and $h = \Delta t = 0.1$.

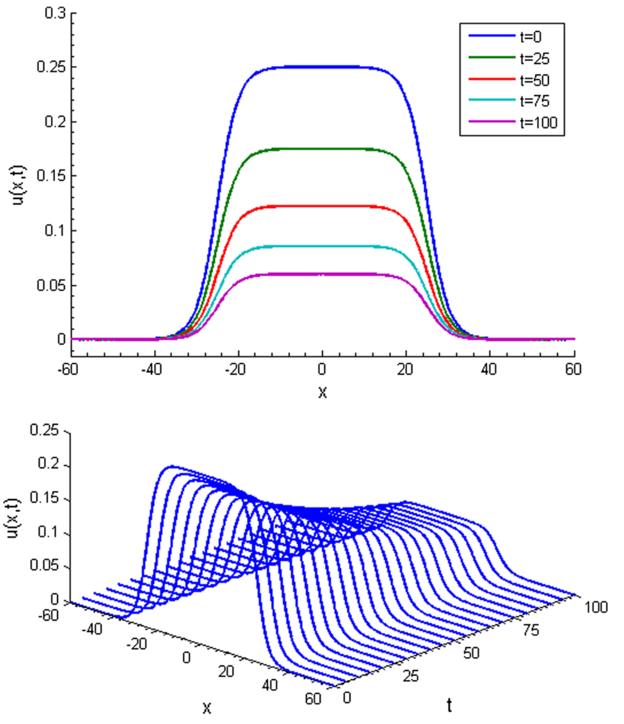

Figure 9. Generated waves profile for $p = 3$, $\alpha = 0.33$, $\beta = 0.1$, $\gamma = 7$, $a = 0.5$ and $h = \Delta t = 0.1$.

As a result, as the time progresses, amplitudes of the waves decrease.

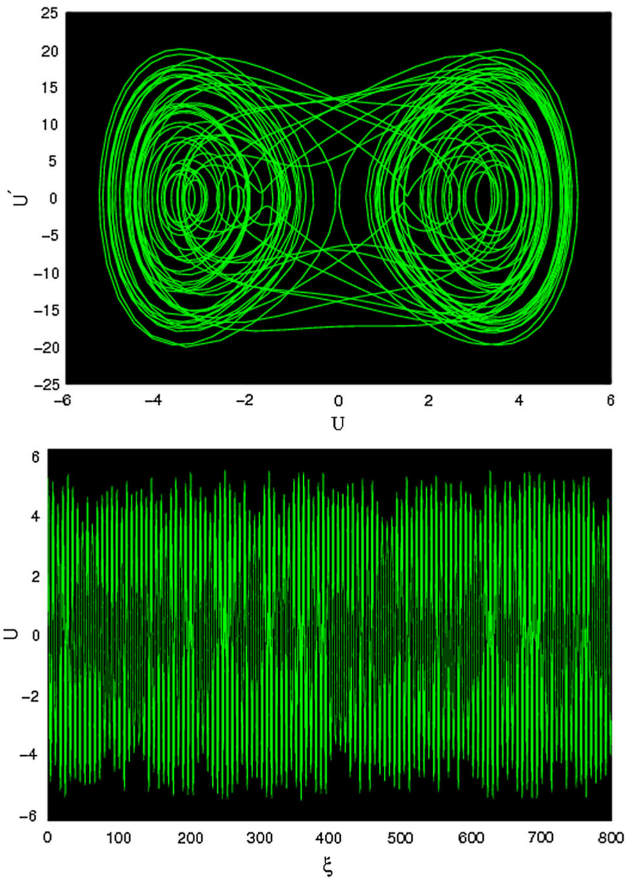

Figure 10. Phase portrait and time series plot of the perturbed generalised Oskolkov system (86) for $p = 3$, $\alpha = 0.3$, $\beta = 0.00215$, $\gamma = -1$, $c = 0.199$, $\omega = 0.9$ and $f_0 = 12.725$.

6. Dynamic motions

Often, there exist different types of periodic perturbations in many real physical situations [27,28]. The nature of these external periodic perturbation may vary depending on different physical situations. Recently, much attention has been paid to the study of nonlinear model equations considering external periodic perturbations [14,18]. In order to study the dynamic motions of the travelling wave solutions of the generalised Oskolkov equation in the presence of an external periodic perturbation, we add an external periodic perturbation $f_0 \cos(\omega\xi)$ to system (15) and obtain the following perturbed system:

$$-cU + \alpha U^p + \beta U' - \gamma c U'' = f_0 \cos(\omega\xi), \tag{86}$$

where f_0 is the strength of the external perturbation and ω is its frequency. Different numerical tools, such as phase portrait plots and time series plots, are applied to investigate the dynamic motions [29–31] of the generalised Oskolkov equation (86) in the presence of

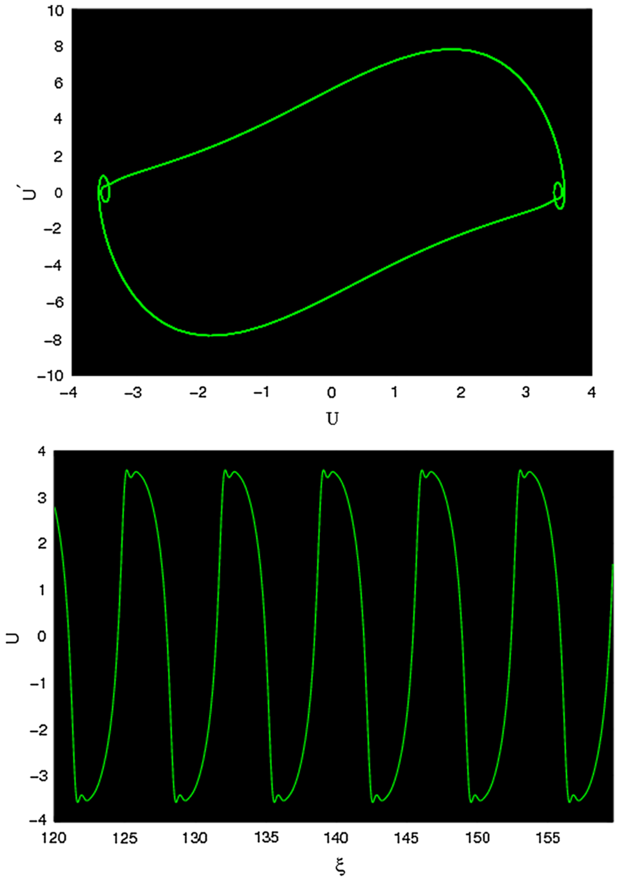

Figure 11. Phase portrait and time series plot of the perturbed generalised Oskolkov system (86) for $\beta = 0.9$. Other parameters are the same as in figure 10.

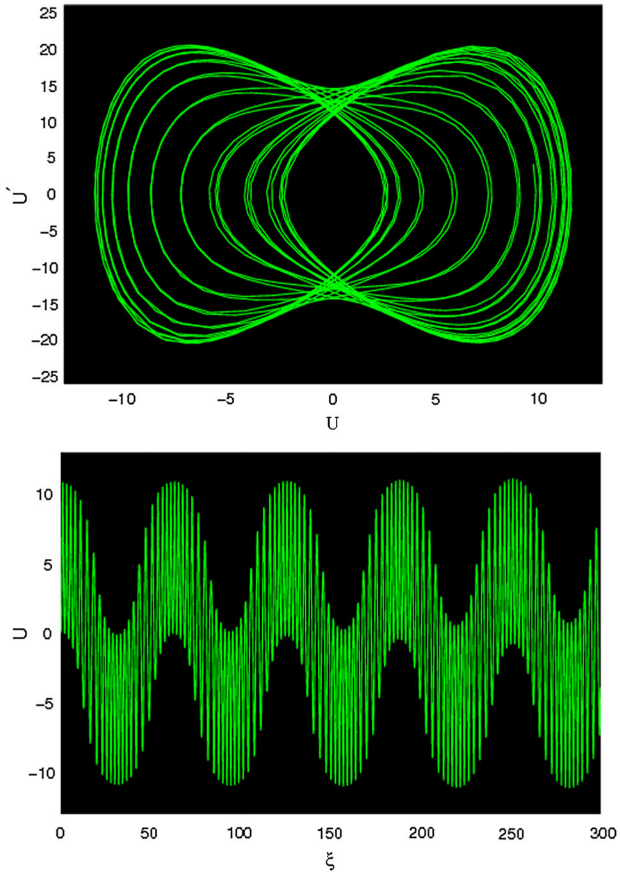

Figure 12. Phase portrait and time series plot of the perturbed generalised Oskolkov system (86) for $c = 3.1, \omega = 0.1$ and $f_0 = 80.725$. Other parameters are the same as in figure 10.

external periodic perturbation depending on the parameters α, β, γ, c, ω and f_0. Because of the large number of parameters, it is difficult to investigate system (86) for the complete range of parametric space. To simplify the task, we focus on some special values of parameters α, γ, c, ω and f_0 varying the parameter β. Thus, β plays a critical role for the dynamical behaviour of the travelling wave solutions of the the generalised Oskolkov equation in the presence of external periodic perturbation. We could vary any of the other parameters, but this will not provide any significant features.

In figure 10, phase portrait and U vs. ξ of the perturbed system (86) are presented for $p = 3$, $\alpha = 0.3$, $\beta = 0.00215$, $\gamma = -1$, $c = 0.199, \omega = 0.9$ and $f_0 = 12.725$. It is found that the orbits presented in the phase portrait ignore periodic motions and exhibit aperiodic random oscillations, which are very sensitive to initial conditions. Thus, the perturbed system (86) shows chaotic motion in this case. If the effect of viscosity is increased and we consider the value of the coefficient of viscosity $\beta = 0.9$ keeping the same values of other parameters as in figure 10, the perturbed system (86) shows period-3 motion, which is depicted in figure 11. Thus, the coefficient of viscosity β plays a critical role in the transition from chaotic motion to period-3 motion for the perturbed system (86). If the velocity c of the travelling wave and the strength f_0 of the external periodic perturbation are increased with low frequency of the external periodic perturbation, the perturbed system (86) ignores the periodic oscillations and exhibits quasiperiodic motions, which is shown in figure 12 for $c = 3.1, \omega = 0.1$ and $f_0 = 80.725$ with the same values for other parameters as in figure 10. Our investigation may be applied to study the nonlinear dynamics of an incompressible viscoelastic Kelvin–Voigt fluid in the presence of an external periodic force [12,13].

7. Conclusion

We have studied the propagation of nonlinear shock waves for the generalised Oskolkov equation and dynamic motions of the perturbed Oskolkov equation.

Employing the unified method, we have derived a collection of exact shock wave solutions for the generalised Oskolkov equations. We have applied collocation finite element method to the generalised Oskolkov equation for checking the accuracy of the proposed method by the two test problems including the motion of shock wave and evolution of waves with Gaussian and undular bore initial conditions. Considering an external periodic perturbation, we have studied the dynamic motions of the perturbed generalised Oskolkov equation depending on the system parameters using the mathematical tools phase portrait and time series analysis. We have shown that the perturbed generalised Oskolkov equation presents period-3, quasiperiodic and chaotic motions for the special values of system parameters. It is seen that in the absence of periodic perturbation, the generalised Oskolkov equation presents shock waves, but it shows period-3, quasiperiodic and chaotic motions when an external periodic perturbation is employed. Our approaches could be applicable for studying nonlinear wave features in diverse fields of nonlinear science such as, fluid mechanics, nuclear physics, solid-state physics, plasma physics, chemical physics, optical fibre and geochemistry.

References

[1] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[2] A Biswas and E Zerrad, *Nonlinear Anal.: Real World Appl.* **11**(**4**), 3272 (2010)
[3] E Yomba, *Chaos Solitons Fractals* **27**(**1**), 187 (2006)
[4] A Biswas and E Zerrad, *Commun. Nonlinear Sci. Numer. Simul.* **14** (**9–10**), 3574 (2009)
[5] M R Miura, *Backlund transformation* (Springer, Berlin, 1978)
[6] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**(**3**), 522 (1983)
[7] S Liu, Z Fu, S Liu and Q Zhao, *Phys. Lett. A* **289** (**1–2**), 69 (2001)
[8] A Biswas, *Appl. Math. Lett.* **22**(**11**), 1775 (2009)
[9] H Junqi, *Chaos Solitons Fractals* **23**(**2**), 391 (2005)
[10] A A Afify and M A El-Aziz, *Pramana – J. Phys.* **88**(**2**), 31 (2017)
[11] A Saha, B Talukdar, U Das and S Chatterjee, *Pramana – J. Phys.* **88**(**2**), 28 (2017)
[12] V B Amfilokhiev, Y I Voitkunskii, N P Mazaeva and Y S Khodorkovskii, *Trudy Leningrad Korablestroit Institute* **96**, 3 (1975)
[13] G A Sviridyuk and A V Ankudinov, *Differ. Equ.* **39**(**11**), 1639 (2003)
[14] F K Abdullaev, *Phys. Rep.* **179**(**1**), 1 (1989)
[15] R Grimshaw and X Tian, *Proc. R. Soc. Lond. A* **445**, 1 (1994)
[16] K B Blyuss, *Rep. Math. Phys.* **49**(**1**), 29 (2002)
[17] D J Zheng, W J Yeh and O G Symko, *Phys. Lett. A* **140**(**5**), 225 (1989)
[18] H T Moon, *Phys. Rev. Lett.* **64**(**4**), 412 (1990)
[19] U K Samanta, A Saha and P Chatterjee, *Phys. Plasmas* **20**, 022111 (2013)
[20] B Sahu, S Poria and R Roychoudhury, *Astrophys. Space Sci.* **341**(**2**), 567 (2012)
[21] Y-Q Zhang, X-Y Wang, L-Y Liu, Y He and J Liu, *Commun. Nonlinear Sci. Numer. Simul.* **52**, 52 (2017)
[22] A Saha, N Pal and P Chatterjee, *Phys. Plasmas* **21**, 102101 (2014)
[23] A S T Nguetcho, J Li and J M Bilbault, *Commun. Nonlinear Sci. Numer. Simul.* **19**(**8**), 2590 (2014)
[24] M M Selim, A El-Depsy and E F El-Shamy, *Astrophys. Space Sci.* **360**, 66 (2015)
[25] O F Gozukizil, S Akcagil and T Aydemir, *Open Phys.* **14**, 524 (2016)
[26] P M Prenter, *Splines and variational methods* (John Wiley, New York, 1975)
[27] A Sen, S Tiwari, S Mishra and P Kaw, *Adv. Space Res.* **56**(**3**), 429 (2015)
[28] Z Jun-Xiao and G Bo-Ling, *Commun. Theor. Phys.* **52**(**2**), 279 (2009)
[29] M Lakshmanan and S Rajasekar, *Nonlinear dynamics* (Springer-Verlag, Berlin, 2003)
[30] A Saha, *Nonlinear Dyn.* **87**(**4**), 2193 (2017)
[31] A Saha, *Comput. Math. Appl.* **73**(**9**), 1879 (2017)