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Some new exact wave solutions and conservation laws of potential Korteweg–de Vries equation

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Abstract The potential Korteweg–de Vries equation is investigated by using the sine–cosine method and generalized Kudryashov method. A variety of exact wave solutions that contain shock solutions, periodic solutions and solitons solutions are formally derived. Also, conservation laws of the equation is determined by using the multiplier approach. In addition, the restrictions on the physical parameters of obtained solutions are presented.

Keywords P-KdV equation · Kudryashov method · Sine–cosine method · Conservation laws

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1 Introduction

Exact travelling wave solutions to nonlinear evolution equations (NLEEs), particularly that appear in many physical structures in solitary wave theory such as solitons, kinks, peakons, and cuspons [1], draw considerable interest in recent years in revealing the mechanism of the complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations [2]. There has been an enormous interest on the study of soliton structures that propagate in nonlinear media without changing their profile by reason of their potential application in many physics areas. Their existence is a result of a delicate balance between linear dispersion and nonlinearity.

Many powerful methods have been developed to find the integrations of NLEEs of all kinds such as the nonlinear Schrödinger equation, the Korteweg–de Vries equation, the sine-Gordon. Among these methods, the subsidiary ordinary differential equation method (sub-ODE for short) [3–5], the homogeneous balance principle and F-expansion method [6], the Jacobi elliptic functions method [7], the sine–cosine and tanh methods [8], the Hirota's bilinear method, the Bäcklund transformation method, and so on, were used. These methods work effectively even though the Painleve test of integrability fails [9]. It is widely believed that possession of the Painlevé property is a sufficient criterion for integrability [10].

The p-KdV equation has been the subject of extensive studies [11–15]. The ansatz method, the G'/G-expansion method, the exp-function approach, mapping method, and the Lie symmetry analysis have mostly been used to investigate this equation. The main purpose of these studies has been the solitons and travelling wave solutions with interesting properties [11–15].

The purpose of this paper is to apply the sine–cosine method and generalized Kudryashov method to construct some exact solutions of distinct physical structures, solitons solutions, periodic wave solutions, and shock wave solutions for the nonlinear dispersive equation. It is significant that this approach gives more general solutions than the ansatz method. The conditions on the physical parameters for the existence of the obtained structures are reported. Also, conservation laws of the equation are determined by using the multiplier approach.

2 Mathematical model

The potential Korteweg–de Vries (p-KdV) equation [11,12]:

$$q_t + a\left(q_x\right)^2 + bq_{xxx} = 0, \tag{1}$$

where a and b real-valued constants, is a nonlinear model of physical importance, especially in the study of water waves. Here in (1), the first term is the evolution term, the second term represents nonlinearity, and the last term is the third-order dispersion term. Also, x and t are the independent variables that represent the spatial and temporal variables, respectively, and $q(x,t)$ is the dependent variable that represents the wave profile.

3 The generalized Kudryashov method

To obtain shock wave solutions of Eq. (1), the wave transformation is used

$$q(x,t) = q(\xi), \quad \xi = x - ct, \tag{2}$$

where c is the wave speed. Under this transformation, Eq. (1) is reduced to the following ordinary differential equation:

$$-cq' + a\left(q'\right)^2 + bq''' = 0. \tag{3}$$

Using generalized Kudryashov method [16–18] in the subsequent section, Eq. (3) is analyzed for shock wave solutions.

To proceed with the solutions of (3), the following assumption for the shock structure is made

$$q(\xi) = \frac{\sum_{i=0}^{N} \alpha_i \Psi^i(\xi)}{\sum_{j=0}^{M} \beta_j \Psi^j(\xi)} = \frac{\alpha_0 + \alpha_1\Psi(\xi) + \cdots + \alpha_N\Psi^N(\xi)}{\beta_0 + \beta_1\Psi(\xi) + \cdots + \beta_M\Psi^M(\xi)}, \tag{4}$$

where α_i $(i = 0, 1, \ldots, N)$, β_j $(j = 0, 1, \ldots, M)$ are constants to be determined later such that $\alpha_N \neq 0$, $\beta_M \neq 0$, and $\Psi = \Psi(\xi)$ satisfy the ODE

$$\frac{d\Psi}{d\xi} = \Psi^2(\xi) - \Psi(\xi). \tag{5}$$

It is well known that Eq. (5) has the following solutions:

$$\Psi(\xi) = \frac{1}{1 \pm e^{\xi}}. \tag{6}$$

Balancing $\left(q'\right)^2$ with q''' in Eq. (3), it is obtained a relation that is given by

$$N = M + 1. \tag{7}$$

Case 1: When $M = 0$ and $N = 1$ in Eq. (7), the solution of Eq. (3) has the form

$$q(\xi) = \frac{\alpha_0 + \alpha_1\Psi(\xi)}{\beta_0}, \tag{8}$$

where $\alpha_1 \neq 0$ and $\beta_0 \neq 0$. Substituting (8) along with Eq. (5) into (3), collecting the coefficients of Ψ, and solving the resulting system, it is recovered

$$\alpha_0 = \alpha_0, \quad \alpha_1 = -\frac{6b\beta_0}{a}, \quad \beta_0 = \beta_0, \quad c = b. \tag{9}$$

As a result, it is obtained exact solutions of the p-KdV equation as follows:

$$q(x,t) = \frac{1}{a\beta_0}\left(a\alpha_0 - 3b\beta_0\left[1 - \tanh\left(\frac{x - bt}{2}\right)\right]\right), \tag{10}$$

and

$$q(x,t) = \frac{1}{a\beta_0}\left(a\alpha_0 - 3b\beta_0\left[1 - \coth\left(\frac{x - bt}{2}\right)\right]\right). \tag{11}$$

Case 2: When $M = 1$ and $N = 2$ in Eq. (7), the solution of Eq. (3) has the form

$$q(\xi) = \frac{\alpha_0 + \alpha_1 \Psi(\xi) + \alpha_2 \Psi^2(\xi)}{\beta_0 + \beta_1 \Psi(\xi)}, \tag{12}$$

where $\alpha_2 \neq 0$ and $\beta_1 \neq 0$. Substituting (12) along with Eq. (5) into (3), collecting the coefficients of Ψ, and solving the resulting system, it is obtained the following results:

Set 1.

$$\begin{aligned} &\alpha_0 = 0, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = -\frac{6b\beta_1}{a}, \\ &\beta_0 = 0, \quad \beta_1 = \beta_1, \quad c = b, \end{aligned} \tag{13}$$

Set 2.

$$\begin{aligned} &\alpha_0 = \alpha_0, \; \alpha_1 = \frac{\alpha_0 \beta_1}{\beta_0} - \frac{6b\beta_0}{a}, \; \alpha_2 = -\frac{6b\beta_1}{a}, \\ &\beta_0 = \beta_0, \; \beta_1 = \beta_1, \; c = b, \end{aligned} \tag{14}$$

Set 3.

$$\begin{aligned} &\alpha_0 = \alpha_0, \quad \alpha_1 = -2\alpha_0, \quad \alpha_2 = \frac{12b\beta_0}{a}, \\ &\beta_0 = \beta_0, \quad \beta_1 = -2\beta_0, \quad c = 4b. \end{aligned} \tag{15}$$

Consequently, it is obtained exact solutions of the p-KdV equation as below:

The solutions of (1) corresponding to (13) are

$$q(x,t) = \frac{1}{a\beta_1}\left(a\alpha_1 - 3b\beta_1\left[1 - \tanh\left(\frac{x - bt}{2}\right)\right]\right), \tag{16}$$

and

$$q(x,t) = \frac{1}{a\beta_1}\left(a\alpha_1 - 3b\beta_1\left[1 - \coth\left(\frac{x - bt}{2}\right)\right]\right). \tag{17}$$

The solutions of (1) corresponding to (14) are

$$q(x,t) = \frac{1}{a\beta_0}\left(a\alpha_0 - 3b\beta_0\left[1 - \tanh\left(\frac{x - bt}{2}\right)\right]\right), \tag{18}$$

and

$$q(x,t) = \frac{1}{a\beta_0}\left(a\alpha_0 - 3b\beta_0\left[1 - \coth\left(\frac{x - bt}{2}\right)\right]\right). \tag{19}$$

The solution of (1) corresponding to (15) is

$$q(x,t) = \frac{1}{a\beta_0}\left(a\alpha_0 - 6b\beta_0\left[1 - \coth\left(x - 4bt\right)\right]\right). \tag{20}$$

Case 3: When $M = 2$ and $N = 3$ in Eq. (7), the solution of Eq. (3) has the form

$$q(\xi) = \frac{\alpha_0 + \alpha_1 \Psi(\xi) + \alpha_2 \Psi^2(\xi) + \alpha_3 \Psi^3(\xi)}{\beta_0 + \beta_1 \Psi(\xi) + \beta_2 \Psi^2(\xi)}, \tag{21}$$

where $\alpha_3 \neq 0$ and $\beta_2 \neq 0$. Substituting (21) along with Eq. (5) into (3), collecting the coefficients of Ψ, and solving the resulting system, it is obtained

Set 1.

$$\begin{aligned} &\alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = -\frac{6b\beta_2}{a}, \\ &\beta_0 = 0, \quad \beta_1 = 0, \quad \beta_2 = \beta_2, \quad c = b, \end{aligned} \tag{22}$$

Set 2.

$$\begin{aligned} &\alpha_0 = 0, \quad \alpha_1 = \frac{\beta_1 \left(a\alpha_2 + 6b\beta_1\right)}{a\beta_2}, \quad \alpha_2 = \alpha_2, \\ &\alpha_3 = -\frac{6b\beta_2}{a}, \beta_0 = 0, \quad \beta_1 = \beta_1, \quad \beta_2 = \beta_2, \quad c = b, \end{aligned} \tag{23}$$

Set 3.

$$\begin{aligned} &\alpha_0 = 0, \quad \alpha_1 = -\frac{3b\beta_1}{a} - \frac{\alpha_2}{2}, \quad \alpha_2 = \alpha_2, \\ &\alpha_3 = \frac{12b\beta_1}{a}, \beta_0 = 0, \quad \beta_1 = \beta_1, \quad \beta_2 = -2\beta_1, \quad c = b, \end{aligned} \tag{24}$$

Set 4.

$$\begin{aligned} &\alpha_0 = \frac{\beta_0 \left(a\alpha_2 + 6b\beta_1\right)}{a\beta_2}, \\ &\alpha_1 = \frac{a\alpha_2\beta_1 + 6b\beta_1^2 - 6b\beta_0\beta_2}{a\beta_2}, \alpha_2 = \alpha_2, \\ &\alpha_3 = -\frac{6b\beta_2}{a}, \beta_0 = \beta_0, \quad \beta_1 = \beta_1, \quad \beta_2 = \beta_2, \quad c = b, \end{aligned} \tag{25}$$

Set 5.

$$\begin{aligned} &\alpha_0 = 0, \quad \alpha_1 = -\frac{\alpha_2}{2}, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \frac{12b\beta_1}{a}, \\ &\beta_0 = 0, \quad \beta_1 = \beta_1, \quad \beta_2 = -2\beta_1, \quad c = 4b, \end{aligned} \tag{26}$$

Set 6.

$$\begin{aligned} &\alpha_0 = \frac{3b\beta_0}{a} - \frac{\alpha_2}{4}, \quad \alpha_1 = 0, \quad \alpha_2 = \alpha_2, \\ &\alpha_3 = \frac{24b\beta_0}{a}, \beta_0 = \beta_0, \quad \beta_1 = 0, \quad \beta_2 = -4\beta_0, \\ &c = 4b, \end{aligned} \tag{27}$$

Set 7.

$$\begin{aligned} &\alpha_0 = -\frac{6b\beta_0}{a}, \quad \alpha_1 = \frac{18b\beta_0}{a}, \quad \alpha_2 = 0, \\ &\alpha_3 = -\frac{12b\beta_0}{a}, \beta_0 = \beta_0, \quad \beta_1 = -3\beta_0, \\ &\beta_2 = 2\beta_0, \quad c = 4b, \end{aligned} \tag{28}$$

Set 8.

$$\begin{aligned}&\alpha_0=\frac{\alpha_2}{2}-\frac{6b\beta_0}{a},\quad \alpha_1=\frac{18b\beta_0}{a}-\frac{3\alpha_2}{2},\\&\alpha_2=\alpha_2,\quad \alpha_3=-\frac{12b\beta_0}{a},\ \beta_0=\beta_0,\\&\beta_1=-3\beta_0,\quad \beta_2=2\beta_0,\quad c=4b,\end{aligned}\tag{29}$$

Set 9.

$$\begin{aligned}&\alpha_0=-\frac{a\alpha_2\beta_0-12b\beta_0^2}{4a\beta_0+2a\beta_1},\quad \alpha_1=-\frac{\beta_1\left(a\alpha_2-12b\beta_0\right)}{2a\left(2\beta_0+\beta_1\right)},\\&\alpha_2=\alpha_2,\quad \alpha_3=\frac{12b\left(2\beta_0+\beta_1\right)}{a},\ \beta_0=\beta_0,\\&\beta_1=\beta_1,\quad \beta_2=-2\left(2\beta_0+\beta_1\right),\quad c=4b,\end{aligned}\tag{30}$$

Set 10.

$$\begin{aligned}&\alpha_0=0,\quad \alpha_1=0,\quad \alpha_2=0,\quad \alpha_3=-\frac{18b\beta_0}{a},\\&\beta_0=\beta_0,\quad \beta_1=-3\beta_0,\quad \beta_2=3\beta_0,\quad c=9b,\end{aligned}\tag{31}$$

Set 11.

$$\begin{aligned}&\alpha_0=\frac{\alpha_2}{3},\quad \alpha_1=-\alpha_2,\quad \alpha_2=\alpha_2,\quad \alpha_3=-\frac{18b\beta_0}{a},\\&\beta_0=\beta_0,\quad \beta_1=-3\beta_0,\quad \beta_2=3\beta_0,\quad c=9b.\end{aligned}\tag{32}$$

As a result, it is obtained exact solutions of the p-KdV equation as follows (see Figs. 1, 2):

The solutions of (1) corresponding to (22) are

$$q(x,t)=\frac{1}{a\beta_2}\left(a\alpha_2-3b\beta_2\left[1-\tanh\left(\frac{x-bt}{2}\right)\right]\right),\tag{33}$$

and

$$q(x,t)=\frac{1}{a\beta_2}\left(a\alpha_2-3b\beta_2\left[1-\coth\left(\frac{x-bt}{2}\right)\right]\right).\tag{34}$$

The solutions of (1) corresponding to (23) and (25) are

$$q(x,t)=\frac{1}{a\beta_2}\left(a\alpha_2+6b\beta_1-3b\beta_2\left[1-\tanh\left(\frac{x-bt}{2}\right)\right]\right),\tag{35}$$

and

$$q(x,t)=\frac{1}{a\beta_2}\left(a\alpha_2+6b\beta_1-3b\beta_2\left[1-\coth\left(\frac{x-bt}{2}\right)\right]\right).\tag{36}$$

The solutions of (1) corresponding to (24) are

$$q(x,t)=-\frac{\alpha_2}{2\beta_1}-\frac{3b}{a}\left[2-\tanh\left(\frac{x-bt}{2}\right)\right],\tag{37}$$

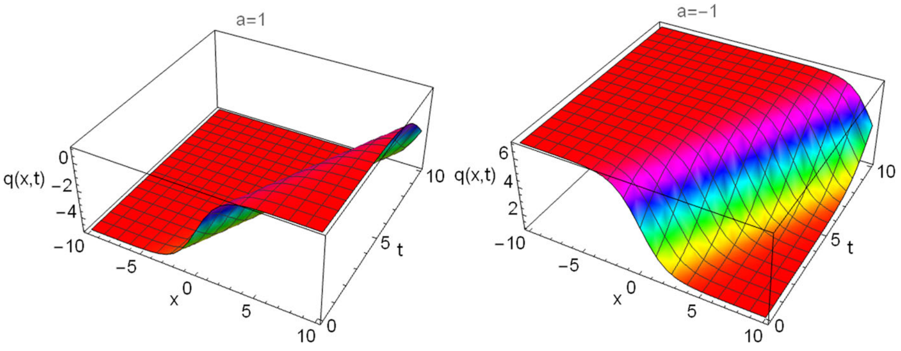

Fig. 1 The 3D surfaces of shock wave solutions of Eq. (1) for different values of a, when $t=0$, $b=1$, and $c=1$

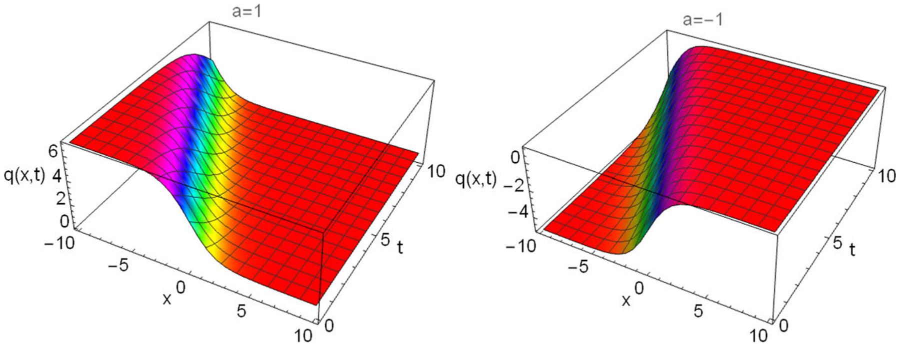

Fig. 2 The 3D surfaces of shock wave solutions of Eq. (1) for different values of a, when $t=0$, $b=-1$, and $c=-1$

and

$$q(x,t) = -\frac{\alpha_2}{2\beta_1} - \frac{3b}{a}\left[2 - \coth\left(\frac{x-bt}{2}\right)\right]. \tag{38}$$

The solution of (1) corresponding to (26) is

$$q(x,t) = -\frac{1}{2a\beta_1}\left(a\alpha_2 + 12b\beta_1\left[1 - \coth\left(x - 4bt\right)\right]\right). \tag{39}$$

The solution of (1) corresponding to (27) is

$$q(x,t) = -\frac{1}{4a\beta_0}\left(a\alpha_2 + 12b\beta_0\left[1 - 2\coth\left(x - 4bt\right)\right]\right). \tag{40}$$

The solution of (1) corresponding to (28) is

$$q(x,t) = -\frac{6b}{a}\left(2 - \coth\left(x - 4bt\right)\right). \tag{41}$$

The solution of (1) corresponding to (29) is

$$q(x,t) = \frac{1}{2a\beta_0}\left(a\alpha_2 - 12b\beta_0\left[2 - \coth\left(x - 4bt\right)\right]\right). \tag{42}$$

The solution of (1) corresponding to (30) is

$$q(x,t) = \frac{1}{2a}\left(\frac{12b\beta_0 - a\alpha_2}{2\beta_0 + \beta_1} - 12b[1 - \coth(x - 4bt)]\right). \tag{43}$$

The solutions of (1) corresponding to (31) are

$$q(x,t) = -\frac{9b\left[1 - \tanh\left(\frac{x-9bt}{2}\right)\right]^3}{a\left[1 + 3\tanh^2\left(\frac{x-9bt}{2}\right)\right]}, \tag{44}$$

and

$$q(x,t) = -\frac{9b\left[1 - \coth\left(\frac{x-9bt}{2}\right)\right]^3}{a\left[1 + 3\coth^2\left(\frac{x-9bt}{2}\right)\right]}. \tag{45}$$

The solutions of (1) corresponding to (32) are

$$q(x,t) = \frac{\alpha_2}{3\beta_0} - \frac{9b}{a}\left[1 - \tanh\left(\frac{3x - 27bt}{2}\right)\right], \tag{46}$$

and

$$q(x,t) = \frac{\alpha_2}{3\beta_0} - \frac{9b}{a}\left[1 - \coth\left(\frac{3x - 27bt}{2}\right)\right]. \tag{47}$$

4 The Sine–Cosine method

Using the wave variable $\xi = x - ct$, where c is the wave speed, the nonlinear equation (1) is converted into

$$-cq_\xi + a\left(q_\xi\right)^2 + bq_{\xi\xi\xi} = 0, \tag{48}$$

where the subscripts denote the derivative of $q(\xi)$ with respect to the travelling variable ξ.

Substituting $q_\xi = Q$ into Eq. (48), it gives

$$-cQ + aQ^2 + bQ_{\xi\xi} = 0, \tag{49}$$

In what follows, the sine–cosine method is employed to extract the exact solutions of the reduced ODE equation (49). It is assumed in the sine–cosine ansatz [1,19] the solutions are in the forms as indicated below:

$$Q(\xi) = \left\{\lambda\cos^\beta(\mu\xi)\right\}, \quad |\xi| \le \frac{\pi}{2\mu} \tag{50}$$

$$Q(\xi) = \left\{\lambda\sin^\beta(\mu\xi)\right\}, \quad |\xi| \le \frac{\pi}{\mu} \tag{51}$$

where λ, μ, and β are parameters that are determined. Here λ and μ represent the wave amplitude and the wave number, respectively. Also, the unknown exponent β is determined using the balancing principle.

Inserting the ansatz (50) into the reduced ODE (49), it gives

$$\begin{aligned} &-c\lambda\cos^\beta(\mu\xi) + a\lambda^2\cos^{2\beta}(\mu\xi) \\ &\quad - b\mu^2\beta^2\lambda\cos^\beta(\mu\xi) \\ &\quad + b\mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0, \end{aligned} \tag{52}$$

and from inserting the ansatz (51), it is obtained

$$\begin{aligned} &-c\lambda\sin^\beta(\mu\xi) + a\lambda^2\sin^{2\beta}(\mu\xi) - b\mu^2\beta^2\lambda\sin^\beta(\mu\xi) \\ &\quad + b\mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) = 0, \end{aligned} \tag{53}$$

Equating the exponents and the coefficients of like powers of cosine functions in (52) leads to

$$\begin{aligned} &\beta(\beta-1) \ne 0, \\ &\beta - 2 = 2\beta, \\ &-c\lambda - b\mu^2\beta^2\lambda = 0, \\ &a\lambda^2 + b\mu^2\lambda\beta(\beta-1) = 0. \end{aligned} \tag{54}$$

Solving this system yields

$$\begin{aligned} &\beta l \ne 0, 1, \\ &\beta = -2, \\ &\mu = \tfrac{1}{2}\sqrt{-\tfrac{c}{b}}, \\ &\lambda = \tfrac{3c}{2a}. \end{aligned} \tag{55}$$

The results (55) give the following periodic solutions to Eq. (49):

$$Q_1(\xi) = \lambda\sec^2(\mu\xi), \tag{56}$$

and

$$Q_2(\xi) = \lambda\csc^2(\mu\xi). \tag{57}$$

where $cb < 0$ and $b \ne 0$. Noting from $q_\xi = Q$, therefore, it is obtained the travelling wave solutions to Eq. (1) as

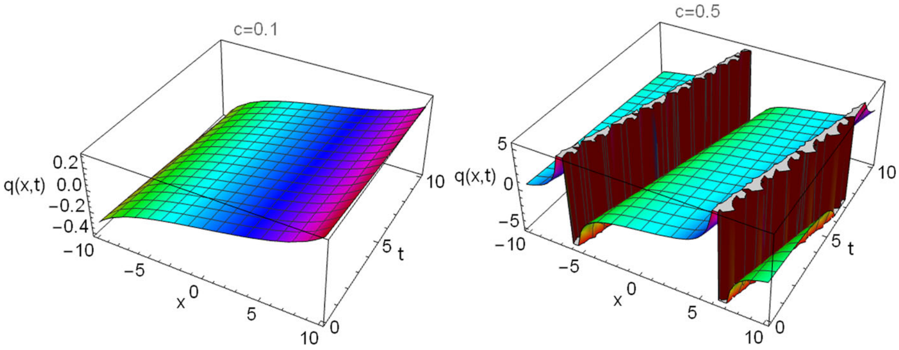

Fig. 3 The $3D$ surfaces of travelling wave solutions of Eq. (1) for different values of c, when $t = 0$, $a = 1$, and $b = -2$

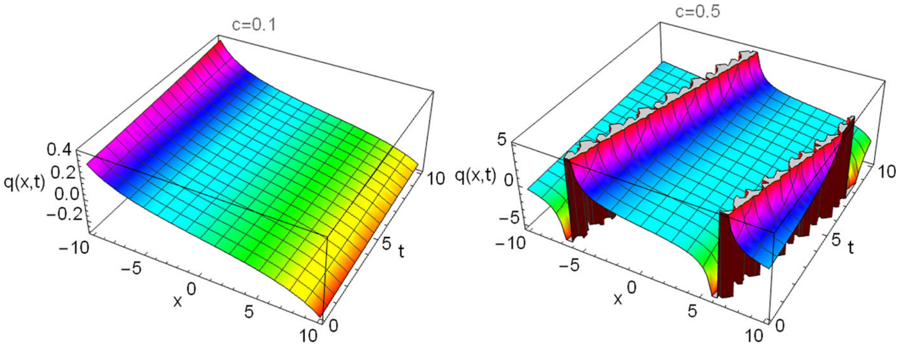

Fig. 4 The $3D$ surfaces of travelling wave solutions of Eq. (1) for different values of c, when $t = 0$, $a = -1$, and $b = -2$

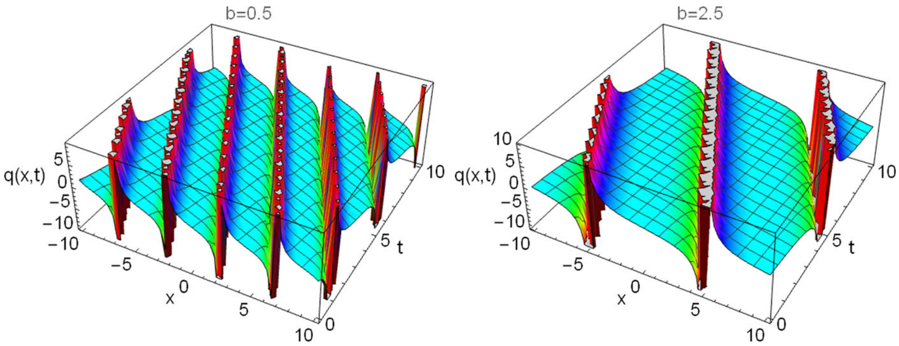

Fig. 5 The $3D$ surfaces of travelling wave solutions of Eq. (1) for different values of b, when $t = 0$, $a = 1$, and $c = -1$

$$q_1(x,t) = \lambda \tan\left[\mu\left(x - ct\right)\right], \tag{58}$$

and

$$q_2(x,t) = \lambda \cot\left[\mu\left(x - ct\right)\right]. \tag{59}$$

However, in the opposite case where $cb > 0$, the exact solutions of Eq. (49) are obtained in the form

$$Q_3(\xi) = \lambda \text{sech}^2\left(\mu\xi\right), \tag{60}$$

and

$$Q_4(\xi) = \lambda \text{csch}^2\left(\mu\xi\right), \tag{61}$$

Integrating Eq. (60) with respect to ξ, it is obtained the topological 1-soliton solution to Eq. (1) as

$$q_3(x,t) = \lambda \tanh\left[\mu\left(x - ct\right)\right], \tag{62}$$

Now, by integrating Eq. (61) with respect to ξ, it is obtained the singular 1-soliton solution to Eq. (1) as

$$q_4(x,t) = \lambda \coth\left[\mu\left(x - ct\right)\right]. \tag{63}$$

Substituting the wave parameters λ and μ given by (55) into the solutions (58), (59), (62) and (63), it can be drawn the figures of (63) as in Figs. 1–6 for some values of a, b, c.

5 Conservation laws

A conserved form of a partial differential equation (PDE)

$$E(x, t, q, q_x, q_t, q_{xx}, \ldots) = 0, \tag{64}$$

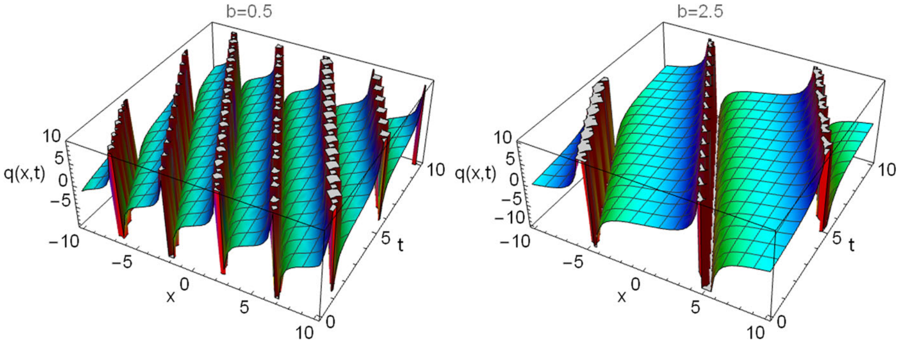

Fig. 6 The 3D surfaces of travelling wave solutions of Eq. (1) for different values of b, when $t = 0$, $a = -1$, and $c = -1$

is a one form $\omega = T^t Dx - T^x Dt$ for which $D\omega$ is a two form which vanishes on the solutions of the PDE (D is the total exterior derivative). It is referred to T^t and T^x as the conserved densities and flux, respectively. These are constructed using the 'multiplier' approach and assuming a second-order derivative dependence of the multipliers $Q = Q(x, t, q, q_x, q_t, q_{xx}, q_{xt}, q_{tt})$. It is obtained the following multipliers Q with conserved forms ω, respectively,

$$
\begin{aligned}
Q_1 &= -q_{xx},\\
Q_2 &= q_{xt},\\
Q_3 &= tq_{xx} - \frac{1}{2a},\\
Q_4 &= u_t q_{xx} + \frac{1}{2a}(-3q_{tt} - 2au_x q_{xt}),
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
\omega_1 = {} & \frac{1}{2}\, qq_{xx} Dx + \left(-\frac{1}{3}\, aq_x{}^3 - \frac{1}{2}\, q_{xx}{}^2 b\right.\\
& \left. -\frac{1}{2}\, q_x q_t + \frac{1}{2}\, qq_{xt}\right) Dt
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\omega_2 = {} & -\frac{1}{2a}\ q\ (tq_{xx}a - 1)\, Dx\\
& + \frac{1}{6a}\Big(2\, ta^2 q_x{}^3 + 3\, q_{xx}{}^2 bta + 3\, taq_x q_t\\
& - 3\, qatq_{xt} - 3\, qaq_x - 3\, q_{xx} b\Big) Dt
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\omega_3 = {} & \left(-\frac{1}{6}\, aq_x{}^3 + \frac{1}{3}\, qq_{xx}aq_x - \frac{1}{4}\, q_x q_t + \frac{1}{4}\, qbq_{xxxx}\right.\\
& \left. -\frac{1}{4}\, q_x bq_{xxx} - \frac{1}{4}\, qq_{xt}\right) Dx\\
& + \left(\frac{1}{6}\, q_t aq_x{}^2 + \frac{1}{3}\, qq_{xt}aq_x + \frac{1}{2}\, q_{xx}q_{xt}b + \frac{1}{4}\, q_t{}^2\right.\\
& + \frac{1}{4}\, q_t bq_{xxx} - \frac{1}{2}\, q_x bq_{xxt} + \frac{1}{4}\, qbq_{xxxt}\\
& \left. -\frac{1}{4}\, qq_{tt}\right) Dt
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\omega_4 = {} & -\frac{1}{24a}\,[-3\, a^2 q_x{}^4 + 15\, qa^2 q_{xx} q_x{}^2 - 16\, q_t aq_x{}^2\\
& + 20\, qq_t q_{xx} a - 4\, aq_x{}^2 bq_{xxx} + 20\, qq_{xt} aq_x\\
& + 12\, qq_{xx} abq_{xxx} + 4\, qq_x bq_{xxxx} a - 18\, q_t{}^2\\
& - 18\, q_t bq_{xxx} + 18\, qbq_{xxxt}] Dx\\
& - \frac{1}{24a}\,[-3\, q_t a^2 q_x{}^3 + 15\, qa^2 q_x{}^2 q_{xt}\\
& - 8\, q_{xx}{}^2 bq_t a + 8\, q_{xx} baq_1 q_{xt} + 8\, qq_{xx} abq_{xxt}\\
& + 4\, q_t aq_x bq_{xxx} - 8\, bq_x{}^2 q_{xxt} a + 20\, qq_t q_{xt} a\\
& + 4\, qbq_{xxx} q_{xt} a + 4\, qq_{xxxt} q_x ba - 4\, q_t{}^2 aq_x\\
& + 20\, qq_x q_{tt} a + 18\, q_{xx} bq_{tt} - 18\, q_x bq_{xtt}\\
& + 18\, qbq_{xxtt}] Dt
\end{aligned}
\tag{69}
$$

so that, in each case,

$$
D\omega = Q(q_t + aq_x^2 + bq_{xxx}) Dx\, Dt. \tag{70}
$$

The point symmetry algebra is spanned by

$$
\begin{aligned}
& X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_q,\\
& X_4 = t_x + \frac{1}{2a} x_q, \quad X_5 = \frac{1}{3} x_x + t_t - \frac{1}{3} q_q
\end{aligned}
\tag{71}
$$

For example, reduction by X_5 leads to the exact solution

$$
q = \frac{1}{4a}\frac{x^2}{t} + k, \tag{72}
$$

where k is a constant.

6 Conclusion

In this study, it has been investigated the potential Korteweg–de Vries equation which arises in the study of water waves. By means of sine–cosine method and generalized Kudryashov method, it has obtained a variety of physical solutions, including periodic solutions, shock solutions, topological and singular soliton solutions. Parametric conditions for the existence of these solutions have been presented. Also, conservation laws of the equation have been determined by using the multiplier approach. The obtained solutions may be useful to explain nonlinear physical phenomena arising in dynamical systems described by the potential Korteweg–de Vries equation.

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