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TRAVELING-WAVE SOLUTION OF THE TZITZÉICA-TYPE EQUATIONS BY USING THE UNIFIED METHOD

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The Tzitzéica equation, the Dodd–Bullough–Mikhailov equation, and the Tzitzéica–Dodd–Bullough equation arise in various branches of science and technology, such as solid state physics, nonlinear dynamics, nonlinear optics, and quantum field theory problems. In this paper, we discuss how to find more general wave solutions of these three nonlinear evolution equations with physical applications using the unified method. The unified method straightforwardly gives many more general solutions with free parameters without the need for extra hardware support. After performing calculations, the graphs of the solutions are plotted using `Maple` *to give insight into the physical structure of the wave solution. Only some selected solutions are plotted to visualize their behavior. Considering the importance of having solutions for nonlinear wave phenomena, we clearly see that the method has many merits and demonstrates a comprehensive application to obtaining solutions in an efficient way. In addition, diverse types of geometrically structured solitons such as an anti-bell-shaped soliton, a flat soliton, a kink, and a singular soliton are produced by using arbitrary parameters.*

Keywords: the unified method, Tzitzéica equation, Dodd–Bullough–Mikhailov equation, Tzitzéica–Dodd–Bullough equation, traveling wave solution

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1. Introduction

Nonlinear partial differential equations (NPDEs) play a significant role in many scientific areas like physics, mathematics, and engineering problems [1]–[5] because many physical phenomena are modeled by them. Therefore, various methods are used to solve partial differential equations (PDEs) in mathematical studies. Some of them are the inverse scattering transform [6], the Bäcklund transform [7], the Darboux transform [8], the Hirota bilinear method [9], the tanh-function method [10], the sine-cosine method [11], the exp-function method [12], the generalized Riccati equation [13], the homogeneous balance method [14], the first-integral method [15], the modified exponential function method [16], [17], the (G'/G)-expansion method [18]–[21], and many others. All these methods are quite successful, but the unified method first proposed by Aydemir in PhD thesis [22], [23] provides more general exact solutions with free parameters A and B in an elegant way [24]–[32]. Besides, the unified method can be applied to both nonlinear partial differential equations and fractional nonlinear partial differential equations [24], [25], [32], and yields a variety of exact solutions of NPDEs in a straightforward, powerful, effective, and reliable way.

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Tzitzéica-type equations are used to solve many solid-state physics, nonlinear optics, and quantum field theory problems. Romanian mathematician Gheorghe Tzitzéica was the first to propose this equation [33], [34], which is important in nonlinear dynamics, to describe special surfaces in differential geometry. Dodd, Bullough, and Mikhailov [35] proposed the Dodd–Bullough–Mikhailov (DBM) and Tzitzéica–Dodd–Bullough (TDB) equations. The most famous of this type of equations is the Tzitzéica equation

$$u_{tt} - u_{xx} - e^{u} + e^{-2u} = 0, \tag{1.1}$$

The second is the DBM equation

$$u_{xt} + e^{u} + e^{-2u} = 0 \tag{1.2}$$

The last one is the TDB equation

$$u_{xt} - e^{-u} - e^{-2u} = 0. \tag{1.3}$$

So far, some different methods have been applied to Tzitzéica-type equations [36]–[39]. By searching through the available academic papers, it has been realized that the unified method, which gives more general solutions of Tzitzéica-type equations, has not been applied yet. Compared with other studies, the main aim of this paper is to examine the Tzitzéica-type equations and find not only the general but also many solutions by using the unified method.

This study is organized as follows. We summarize the unified method in Sec. 2. The application of the method to the equations is presented in Sec. 3. Physical structures and graphical illustrations of the solutions are given in Sec. 4. Our conclusions are given in Sec. 5.

2. Description of the unified method

In this section, we describe the unified method in several steps [23]. We assume that an NPDE in two independent variables x and t is given in the form

$$P(v, v_t, v_x, v_{xt}, v_{tt}, v_{xx}, \dots) = 0, \tag{2.1}$$

where $v(x,t)$ is an unknown function and P is an equation for $v = v(x,t)$ and its various partial derivatives, where the highest-order derivative and nonlinear terms are involved.

Step 1: The wave variable η is used to find the traveling-wave solutions of Eq. (2.1),

$$v(x,t) = V(\eta), \qquad \eta = x - ct, \tag{2.2}$$

where the parameter c is generally termed the wave velocity. Substituting Eq. (2.2) in (2.1) gives an ordinary differential equation (ODE) for η,

$$P(V, cV', V', cV'', c^2V'', V'', \dots) = 0. \tag{2.3}$$

For simplicity, Eq. (2.3) is to be integrated as many times as possible.

Step 2: Suppose a solution of the NPDE can be expressed by the ansatz

$$V(\eta) = a_0 + \sum_{m=1}^{M} [a_m \phi^m + b_m \phi^{-m}], \tag{2.4}$$

where $\phi = \phi(\eta)$ satisfies the Riccati differential equation

$$\phi'(\eta) = \mu + \phi^2(\eta), \tag{2.5}$$

where $\phi' = d\phi/d\eta$, a_m and b_m are coefficients of ϕ, and μ is a parameter. Considering the homogeneous balance between the highest-order derivatives and nonlinear terms in Eq. (2.3), the positive integer M called the balancing term is calculated as explained in step 3.

Family 1. If $\mu < 0$, the solutions of Eq. (2.5) is

$$\phi(\eta) = \begin{cases} \phi_1 = \dfrac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}, \\ \phi_2 = \dfrac{-\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}, \\ \phi_3 = \sqrt{-\mu} + \dfrac{-2A\sqrt{-\mu}}{A + \cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}, \\ \phi_4 = -\sqrt{-\mu} + \dfrac{2A\sqrt{-\mu}}{A + \cosh(2\sqrt{-\mu}(\eta+\eta_0)) + \sinh(2\sqrt{-\mu}(\eta+\eta_0))}. \end{cases} \tag{2.6}$$

Family 2. If $\mu > 0$, the solutions of Eq. (2.5) is

$$\phi(\eta) = \begin{cases} \phi_5 = \dfrac{\sqrt{(A^2-B^2)\mu} - A\sqrt{\mu}\cos(2\sqrt{\mu}(\eta+\eta_0))}{A\sin(2\sqrt{\mu}(\eta+\eta_0)) + B}, \\ \phi_6 = \dfrac{-\sqrt{(A^2-B^2)\mu} - A\sqrt{\mu}\cos(2\sqrt{\mu}(\eta+\eta_0))}{A\sin(2\sqrt{\mu}(\eta+\eta_0)) + B}, \\ \phi_7 = i\sqrt{\mu} + \dfrac{-2Ai\sqrt{\mu}}{A + \cos(2\sqrt{\mu}(\eta+\eta_0)) - i\sin(2\sqrt{\mu}(\eta+\eta_0))}, \\ \phi_8 = -i\sqrt{\mu} + \dfrac{2Ai\sqrt{\mu}}{A + \cos(2\sqrt{\mu}(\eta+\eta_0)) + i\sin(2\sqrt{\mu}(\eta+\eta_0))}, \end{cases} \tag{2.7}$$

where $A \neq 0$ and B are real arbitrary parameters, and η_0 is an arbitrary parameter.

Family 3. If $\mu = 0$, the solution of Eq. (2.5) is

$$\phi(\eta) = -\frac{1}{\eta+\eta_0}, \tag{2.8}$$

where η_0 is an arbitrary parameter.

We note that using the relations $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$, family 2 can be produced from family 1 smoothly. In other words, the hyperbolic solutions in Eq. (2.6) can easily be transformed into trigonometric solutions in Eq. (2.7) when considering trigonometric–hyperbolic relations and changing the sign of μ.

Step 3: The balancing term M can be determined by considering the homogeneous balance between the linear term of the highest order with the nonlinear term of highest degree appearing in Eq. (2.3) as follows. If it is defined as the degree of $V(\eta)$, $D[V(\eta)] = M$, then the degree of other expressions is

$$D\left[\frac{d^qV}{d\eta^q}\right] = M + q, \tag{2.9}$$

$$D\left[V^r\left(\frac{d^qV}{d\eta^q}\right)^s\right] = Mr + s(q+M). \tag{2.10}$$

Step 4: Substituting Eqs. (2.4) and (2.5) in Eq. (2.3), collecting all terms with like powers of ϕ in the final equation, and equating the coefficients at each powers of ϕ to zero, we obtain a set of algebraic equations for a_m, b_m, c, and μ.

Step 5: Substituting a_m, b_m, c, and μ obtained from step 4 in Eq. (2.4) and using the general solutions of Eq. (2.5) given in Eqs. (2.6), (2.7), and (2.8), we obtain the exact solutions of Eq. (2.1) in closed form.

3. Application

3.1. The Tzitzéica equation. The Tzitzéica equation

$$u_{tt} - u_{xx} - e^u + e^{-2u} = 0 \tag{3.1}$$

which has many applications in quantum field theory, nonlinear optics, and solid state physics. The unified method is not applied directly to the Tzitzéica equation because of the exponential terms. Hence, this equation is transformed as

$$vv_{tt} - vv_{xx} - v_t^2 + v_x^2 - v^3 + 1 = 0, \tag{3.2}$$

by using the transformation $u = \ln v$ for the unified method to be applied easily. Using the wave variable $\eta = x - ct$ in Eq. (3.2), we then have

$$(c^2 - 1)VV'' - (c^2 - 1)V'^2 - V^3 + 1 = 0, \tag{3.3}$$

where $v(x,t) = V(\eta)$ and $\eta = x - ct$. Balancing the linear term of the highest order V'^2 with the nonlinear term of the highest degree V^3 gives this simple equation $2M + 2 = 3M$. It hence follows that $M = 2$. Therefore, the solutions of Eq. (3.2) can be written in the form

$$V(\eta) = a_0 + a_1\phi + a_2\phi^2 + \frac{b_1}{\phi} + \frac{b_2}{\phi^2}, \tag{3.4}$$

where a_0, a_1, a_2, b_1, and b_2 are coefficients of ϕ, to be determined later. Substituting Eq. (3.4) and its derivatives in Eq. (3.3) and equating each coefficients of ϕ to zero gives a set of nonlinear algebraic equations for a_0, a_1, a_2, b_1, b_2, and c. Solving these algebraic equations by using `Maple`, we obtained the following sets of parameters:

1. $a_1 = a_2 = b_1 = 0, \quad a_0 = -\dfrac{1}{2}, \quad b_2 = -\dfrac{3\mu}{2}, \quad c = \sqrt{1 - \dfrac{3}{4\mu}}$;

2. $b_1 = b_2 = a_1 = 0, \quad a_0 = -\dfrac{1}{2}, \quad a_2 = -\dfrac{3}{2\mu}, \quad c = \sqrt{1 - \dfrac{3}{4\mu}}$;

3. $b_1 = a_1 = 0, \quad a_0 = \dfrac{1}{4}, \quad a_2 = -\dfrac{3}{8\mu}, \quad b_2 = -\dfrac{3\mu}{8}, \quad c = \sqrt{1 - \dfrac{3}{16\mu}}$;

4. $a_1 = a_2 = b_1 = 0, \quad a_0 = \dfrac{b_2}{3\mu}, \quad b_2 = \dfrac{3\mu}{2}\left(\dfrac{1 - i\sqrt{3}}{2}\right), \quad c = \sqrt{1 + \dfrac{b_2}{2\mu^2}}$;

5. $b_1 = b_2 = a_1 = 0, \quad a_0 = \dfrac{a_2\mu}{3}, \quad a_2 = \dfrac{3}{4\mu}(1 - i\sqrt{3}), \quad c = \sqrt{1 + \dfrac{a_2}{2}}$;

6. $a_1 = a_2 = b_1 = 0, \quad a_0 = \dfrac{b_2}{3\mu}, \quad b_2 = \dfrac{3\mu}{2}\left(\dfrac{1 + i\sqrt{3}}{2}\right), \quad c = \sqrt{1 + \dfrac{b_2}{2\mu^2}}$;

7. $b_1 = b_2 = a_1 = 0, \quad a_0 = \dfrac{a_2\mu}{3}, \quad a_2 = \dfrac{3}{4\mu}(1 + i\sqrt{3}), \quad c = \sqrt{1 + \dfrac{a_2}{2}}$;

8. $b_1 = a_1 = 0, \quad a_0 = -\dfrac{2b_2}{3\mu}, \quad a_2 = \dfrac{b_2}{\mu^2}, \quad b_2 = \dfrac{3\mu}{16}(1 + i\sqrt{3}), \quad c = \sqrt{1 + \dfrac{b_2}{2\mu^2}}$;

9. $b_1 = a_1 = 0, \quad a_0 = -\dfrac{2b_2}{3\mu}, \quad a_2 = \dfrac{b_2}{\mu^2}, \quad b_2 = \dfrac{3\mu}{16}(1 - i\sqrt{3}), \quad c = \sqrt{1 + \dfrac{b_2}{2\mu^2}}$.

Throughout this paper, the solutions are evaluated in the same manner as follows: $u_{n,m}$ implies that the first index n indicates the set number above and the second index m shows which of the ϕ solutions is used in the described solution. For example, the $u_{1,3}$ solution comes from Set 1 and ϕ_3. Substituting the values of the set and ϕ solution in Eq. (3.4), we obtain the exact solutions of the Tzitzéica equation

$$u_{1,1}(x,t) = \ln\left[-\frac{1}{2} - \frac{3\mu}{2}\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\right], \tag{3.5}$$

$$u_{1,3}(x,t) = \ln\left[-\frac{1}{2} - \frac{3\mu/2}{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}\right], \tag{3.6}$$

$$u_{2,1}(x,t) = \ln\left[-\frac{1}{2} - \frac{3}{2\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right)^2\right], \tag{3.7}$$

$$u_{2,3}(x,t) = \ln\left[-\frac{1}{2} - \frac{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{2\mu/3}\right], \tag{3.8}$$

where $\eta = x - \sqrt{1 - 3t/(4\mu)}$,

$$\begin{aligned} u_{3,1}(x,t) = \ln\Bigg[&\frac{1}{4} - \frac{3\mu}{8}\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2 - \\ &- \frac{3}{8\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right)^2\Bigg], \end{aligned} \tag{3.9}$$

$$\begin{aligned} u_{3,3}(x,t) = \ln\Bigg[&\frac{1}{4} - \frac{3\mu/8}{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2} - \\ &- \frac{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{8\mu/3}\Bigg], \end{aligned} \tag{3.10}$$

where $\eta = x - \sqrt{1 - 3t/(16\mu)}$,

$$\begin{aligned} u_{4,1}(x,t) = \ln\Bigg[&\frac{1-i\sqrt{3}}{4} + \frac{3\mu(1-i\sqrt{3})}{4} \times \\ &\times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \end{aligned} \tag{3.11}$$

$$u_{4,3}(x,t) = \ln\left[\frac{1-i\sqrt{3}}{4} + \frac{3\mu(1-i\sqrt{3})/4}{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}\right], \tag{3.12}$$

$$\begin{aligned} u_{5,1}(x,t) = \ln\Bigg[&\frac{1-i\sqrt{3}}{4} + \frac{3(1-i\sqrt{3})}{4\mu} \times \\ &\times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \end{aligned} \tag{3.13}$$

$$\begin{aligned} u_{5,3}(x,t) = \ln\Bigg[&\frac{1-i\sqrt{3}}{4} + \frac{3(1-i\sqrt{3})}{4\mu} \times \\ &\times \left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \end{aligned} \tag{3.14}$$

where $\eta = x - \sqrt{1 + 3t(1 - i\sqrt{3})/(8\mu)}$,

$$u_{6,1}(x,t) = \ln\Bigg[\frac{1+i\sqrt{3}}{4} + \frac{3\mu(1+i\sqrt{3})}{4} \times \\ \times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \tag{3.15}$$

$$u_{6,3}(x,t) = \ln\Bigg[\frac{1+i\sqrt{3}}{4} + \frac{3\mu(1+i\sqrt{3})/4}{(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))})^2}\Bigg], \tag{3.16}$$

$$u_{7,1}(x,t) = \ln\Bigg[\frac{1+i\sqrt{3}}{4} + \frac{3(1+i\sqrt{3})}{4\mu} \times \\ \times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \tag{3.17}$$

$$u_{7,3}(x,t) = \ln\Bigg[\frac{1+i\sqrt{3}}{4} + \frac{3(1+i\sqrt{3})}{4\mu} \times \\ \times \left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \tag{3.18}$$

where $\eta = x - \sqrt{1 + 3t(1 + i\sqrt{3})/(8\mu)}$,

$$u_{8,1}(x,t) = \ln\Bigg[-\frac{1+i\sqrt{3}}{8} + \frac{3\mu(1+i\sqrt{3})}{16} \times \\ \times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2 + \\ + \frac{3(1+i\sqrt{3})}{16\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right)^2\Bigg], \tag{3.19}$$

$$u_{8,3}(x,t) = \ln\Bigg[-\frac{1+i\sqrt{3}}{8} + \frac{3\mu(1+i\sqrt{3})/16}{(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))})^2} - \\ - \frac{(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))})^2}{16\mu/3(1+i\sqrt{3})}\Bigg], \tag{3.20}$$

where $\eta = x - \sqrt{1 + 3t(1 + i\sqrt{3})/(32\mu)}$,

$$u_{9,1}(x,t) = \ln\Bigg[-\frac{1-i\sqrt{3}}{8} + \frac{3\mu(1-i\sqrt{3})}{16} \times \\ \times \left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2 + \\ + \frac{3(1-i\sqrt{3})}{16\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right)^2\Bigg], \tag{3.21}$$

$$u_{9,3}(x,t) = \ln\Bigg[-\frac{1-i\sqrt{3}}{8} + \frac{3\mu(1-i\sqrt{3})/(16)}{(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))})^2} - \\ - \frac{(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))})^2}{16\mu/3(1-i\sqrt{3})}\Bigg], \tag{3.22}$$

where $\eta = x - \sqrt{1 + 3t(1 - i\sqrt{3})/(32\mu)}$.

Here, $\mu < 0$, $A \neq 0$, B, and η_0 are arbitrary parameters. Obviously, $u_{1,1}(x,t) = u_{1,2}(x,t)$ and $u_{1,3}(x,t) = u_{1,4}(x,t)$ give the same solution because squaring ϕ_1 and ϕ_2 or ϕ_3 and ϕ_4 is the same. That is why the repetitive solutions have not been written.

Using the relations $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$ with $\mu > 0$ in Eq. (2.6) yields the solutions in trigonometric form, like that in Eq. (2.7). Therefore, the solutions given above can be converted to trigonometric functions using ϕ solutions in Eq. (2.7).

3.2. The DBM equation. The DBM equation is given by

$$u_{xt} + e^{u} + e^{-2u} = 0. \tag{3.23}$$

By following the same steps as in the preceding subsection, the solutions of the DBM equation can be found. To easily apply the unified method, the transformation $u = \ln v$ is used in Eq. (3.23):

$$vv_{xt} - v_x v_t + v^3 + 1 = 0. \tag{3.24}$$

Using the wave variable $\eta = x - ct$ in Eq. (3.24) gives

$$-cVV'' + cV'^2 + V^3 + 1 = 0, \tag{3.25}$$

where $v(x,t) = V(\eta)$ and $\eta = x - ct$. Balancing V'^2 with V^3 gives $2M + 2 = 3M$, whence $M = 2$. The solutions of Eq. (3.24) can be written in the form

$$V(\eta) = a_0 + a_1\phi + a_2\phi^2 + \frac{b_1}{\phi} + \frac{b_2}{\phi^2}, \tag{3.26}$$

where a_0, a_1, a_2, b_1 and b_2 are coefficients of ϕ, which are determined similarly. The obtained parameter sets are given by

1. $a_1 = a_2 = b_1 = 0, \quad a_0 = \dfrac{1}{2}, \quad b_2 = \dfrac{3\mu}{2}, \quad c = \dfrac{3}{4\mu};$
2. $b_1 = b_2 = a_1 = 0, \quad a_0 = \dfrac{1}{2}, \quad a_2 = \dfrac{3}{2\mu}, \quad c = \dfrac{3}{4\mu};$
3. $b_1 = a_1 = 0, \quad a_0 = -\dfrac{1}{4}, \quad a_2 = \dfrac{3}{8\mu}, \quad b_2 = \dfrac{3\mu}{8}, \quad c = \dfrac{3}{16\mu};$
4. $a_1 = a_2 = b_1 = 0, \quad a_0 = \dfrac{b_2}{3\mu}, \quad b_2 = -\dfrac{3\mu}{4}(1 + i\sqrt{3}), \quad c = \dfrac{b_2}{2\mu^2};$
5. $b_1 = b_2 = a_1 = 0, \quad a_0 = \dfrac{a_2\mu}{3}, \quad a_2 = -\dfrac{3}{4\mu}(1 + i\sqrt{3}), \quad c = \dfrac{a_2}{2};$
6. $a_1 = a_2 = b_1 = 0, \quad a_0 = \dfrac{b_2}{3\mu}, \quad b_2 = -\dfrac{3\mu}{4}(1 - i\sqrt{3}), \quad c = \dfrac{b_2}{2\mu^2};$
7. $b_1 = b_2 = a_1 = 0, \quad a_0 = \dfrac{a_2\mu}{3}, \quad a_2 = -\dfrac{3}{4\mu}(1 - i\sqrt{3}), \quad c = \dfrac{a_2}{2};$
8. $b_1 = a_1 = 0, \quad a_0 = -\dfrac{2b_2}{3\mu}, \quad a_2 = \dfrac{b_2}{\mu^2}, \quad b_2 = -\dfrac{3\mu}{16}(1 - i\sqrt{3}), \quad c = \dfrac{b_2}{2\mu^2};$
9. $b_1 = a_1 = 0, \quad a_0 = -\dfrac{2b_2}{3\mu}, \quad a_2 = \dfrac{b_2}{\mu^2}, \quad b_2 = -\dfrac{3\mu}{16}(1 + i\sqrt{3}), \quad c = \dfrac{b_2}{2\mu^2}.$

Using these values, the following exact solutions of the DBM equation are obtained:

$$u_{1,1}(x,t)=\ln\left[\frac{1}{2}+\frac{3\mu}{2}\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\right], \tag{3.27}$$

$$u_{1,3}(x,t)=\ln\left[\frac{1}{2}+\frac{3\mu/2}{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}\right], \tag{3.28}$$

$$u_{2,1}(x,t)=\ln\left[\frac{1}{2}+\frac{3}{2\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\right], \tag{3.29}$$

$$u_{2,3}(x,t)=\ln\left[\frac{1}{2}+\frac{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{2\mu/3}\right], \tag{3.30}$$

where $\eta=x-3t/4\mu$,

$$\begin{aligned}u_{3,1}(x,t)=\ln\Bigg[&-\frac{1}{4}+\frac{3\mu}{8}\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2+\\&+\frac{3}{8\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\Bigg],\end{aligned} \tag{3.31}$$

$$\begin{aligned}u_{3,3}(x,t)=\ln\Bigg[&-\frac{1}{4}+\frac{3\mu/8}{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}+\\&+\frac{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{8\mu/3}\Bigg],\end{aligned} \tag{3.32}$$

where $\eta=x-3t/16\mu$,

$$\begin{aligned}u_{4,1}(x,t)=\ln\Bigg[&-\frac{1+i\sqrt{3}}{4}-\frac{3\mu(1+i\sqrt{3}\,)}{4}\times\\&\times\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg],\end{aligned} \tag{3.33}$$

$$u_{4,3}(x,t)=\ln\left[-\frac{1+i\sqrt{3}}{4}-\frac{3\mu(1+i\sqrt{3}\,)/4}{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}\right], \tag{3.34}$$

$$\begin{aligned}u_{5,1}(x,t)=\ln\Bigg[&-\frac{1+i\sqrt{3}}{4}-\frac{3(1+i\sqrt{3}\,)}{4\mu}\times\\&\times\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\Bigg],\end{aligned} \tag{3.35}$$

$$\begin{aligned}u_{5,3}(x,t)=\ln\Bigg[&-\frac{1+i\sqrt{3}}{4}-\frac{3(1+i\sqrt{3}\,)}{4\mu}\times\\&\times\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg],\end{aligned} \tag{3.36}$$

where $\eta=x+3(1+i\sqrt{3}\,)t/8\mu$,

$$u_{6,1}(x,t)=\ln\Bigg[-\frac{1-i\sqrt{3}}{4}-\frac{3\mu(1-i\sqrt{3}\,)}{4}\times$$
$$\times\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \tag{3.37}$$

$$u_{6,3}(x,t)=\ln\left[-\frac{1-i\sqrt{3}}{4}-\frac{3\mu(1-i\sqrt{3}\,)/4}{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}\right], \tag{3.38}$$

$$u_{7,1}(x,t)=\ln\Bigg[-\frac{1-i\sqrt{3}}{4\mu}-\frac{1-i\sqrt{3}}{4\mu}\times$$
$$\times\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\Bigg], \tag{3.39}$$

$$u_{7,3}(x,t)=\ln\Bigg[-\frac{1-i\sqrt{3}}{4\mu}-\frac{1-i\sqrt{3}}{4\mu}\times$$
$$\times\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2\Bigg], \tag{3.40}$$

where $\eta=x+3(1-i\sqrt{3}\,)t/8\mu$,

$$u_{8,1}(x,t)=\ln\Bigg[\frac{1-i\sqrt{3}}{8}-\frac{3\mu(1-i\sqrt{3}\,)}{16}\times$$
$$\times\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2-$$
$$-\frac{3(1-i\sqrt{3}\,)}{16\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\Bigg], \tag{3.41}$$

$$u_{8,3}(x,t)=\ln\Bigg[\frac{1-i\sqrt{3}}{8}-\frac{3\mu(1-i\sqrt{3}\,)}{16}\times$$
$$\times\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^{-2}-$$
$$-\frac{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{16\mu/3(1-i\sqrt{3}\,)}\Bigg], \tag{3.42}$$

where $\eta=x+3(1-i\sqrt{3}\,)t/32\mu$,

$$u_{9,1}(x,t)=\ln\Bigg[\frac{1+i\sqrt{3}}{8}-\frac{3\mu(1+i\sqrt{3}\,)}{16}\times$$
$$\times\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2-$$
$$-\frac{3(1+i\sqrt{3}\,)}{16\mu}\left(\frac{\sqrt{-(A^2+B^2)\mu}-A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0))+B}\right)^2\Bigg], \tag{3.43}$$

$$u_{9,3}(x,t)=\ln\Bigg[\frac{1+i\sqrt{3}}{8}-\frac{3\mu(1+i\sqrt{3}\,)/16}{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}-$$
$$-\frac{\left(\sqrt{-\mu}-\frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)^2}{16\mu/3(1+i\sqrt{3}\,)}\Bigg], \tag{3.44}$$

where $\eta=x+3(1+i\sqrt{3}\,)t/32\mu$.

3.3. The TDB equation. The TDB equation is given by

$$u_{xt} - e^{-u} - e^{-2u} = 0. \tag{3.45}$$

Likewise, Eq. (3.45) is transformed by using the transformation $u = -\ln v$ to apply the unified method easily, with the result

$$vv_{xt} - v_x v_t - v^3 - v^4 = 0. \tag{3.46}$$

Using the wave variable $\eta = x - ct$ in Eq. (3.46), we obtain

$$-cVV'' + cV'^2 - V^3 - V^4 = 0, \tag{3.47}$$

where $v(x,t) = V(\eta)$ and $\eta = x - ct$. Balancing V'^2 with V^4 gives $2M + 2 = 4M$, whence $M = 1$. The solutions of Eq. (3.46) can be written in the form

$$V(\eta) = a_0 + a_1\phi + \frac{b_1}{\phi}, \tag{3.48}$$

where a_0, a_1 and b_1 are coefficients of ϕ, which are determined similarly. The obtained parameter sets are given by

1. $a_0 = -\frac{1}{2}, \quad a_1 = 0, \quad b_1 = -\frac{\sqrt{-\mu}}{2}, \quad c = \frac{1}{4\mu};$
2. $a_0 = -\frac{1}{2}, \quad a_1 = 0, \quad b_1 = \frac{\sqrt{-\mu}}{2}, \quad c = \frac{1}{4\mu};$
3. $a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{2\sqrt{-\mu}}, \quad b_1 = 0, \quad c = \frac{1}{4\mu};$
4. $a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2\sqrt{-\mu}}, \quad b_1 = 0, \quad c = \frac{1}{4\mu};$
5. $a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{4\sqrt{-\mu}}, \quad b_1 = \frac{1}{16a_1}, \quad c = \frac{1}{16\mu};$
6. $a_0 = -\frac{1}{2}, \quad a_1 = -\frac{1}{4\sqrt{-\mu}}, \quad b_1 = \frac{1}{16a_1}, \quad c = \frac{1}{16\mu}.$

Using these values, the following exact solutions of the TDB equation are obtained:

$$u_{1,1}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{\sqrt{-\mu}}{2}\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right], \tag{3.49}$$

$$u_{1,3}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{\sqrt{-\mu}/2}{\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}}\right], \tag{3.50}$$

$$u_{3,1}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{1}{2\sqrt{-\mu}}\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right], \tag{3.51}$$

$$u_{3,3}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0))-\sinh(2\sqrt{-\mu}(\eta+\eta_0))}}{2\sqrt{-\mu}}\right], \tag{3.52}$$

where $\eta = x - t/4\mu$,

$$u_{5,1}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{\sqrt{-\mu}}{4}\left(\frac{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}\right) \pm \right.$$
$$\left. \pm \frac{1}{4\sqrt{-\mu}}\left(\frac{\sqrt{-(A^2+B^2)\mu} - A\sqrt{-\mu}\cosh(2\sqrt{-\mu}(\eta+\eta_0))}{A\sinh(2\sqrt{-\mu}(\eta+\eta_0)) + B}\right)\right], \tag{3.53}$$

$$u_{5,3}(x,t) = -\ln\left[-\frac{1}{2} \pm \frac{\sqrt{-\mu}/4}{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)} \pm \right.$$
$$\left. \pm \frac{\left(\sqrt{-\mu} - \frac{2A\sqrt{-\mu}}{A+\cosh(2\sqrt{-\mu}(\eta+\eta_0)) - \sinh(2\sqrt{-\mu}(\eta+\eta_0))}\right)}{4\sqrt{-\mu}}\right], \tag{3.54}$$

where $\eta = x - t/16\mu$.

4. Results and discussion

In this section, the solutions obtained by the unified method are compared with the solutions obtained previously [36]–[39] for Tzitzéica type equations and then the physical structures of some selected solutions are plotted by `Maple`.

Obtaining more general solutions than by other methods without recourse to a tedious and exhaustive computer algorithm was demonstrated in Sec. 3 by using the unified method. In [40], we proved that the unified method gives more solutions than the family of the (G'/G)-expansion method. Because of providing more solution sets and also improving the solution type by adding the term $b_m\phi^{-m}$, the number of the obtained solutions in this study is greater than in [37], [39], where the (G'/G)-expansion method was used.

Furthermore, we also proved that the unified method gives more solutions than the family of the tanh method in [23]. To show that the solutions obtained by applying the tanh method can also be found by applying the unified method, we substitute $B = 0$ and use the equalities $\cosh(2x) = 1 - 2\sinh^2(x)$ and $\sinh(2x) = 2\sinh(x)\cosh(x)$ in family 1. Therefore, the number of solutions obtained by the unified method is also greater than in papers [36], [38], where the tanh method was used.

Only selected solutions of the Tzitzéica equation in Sec. 3.1 are plotted here. These solutions are $u_{1,1}$, $u_{1,3}$, $u_{3,1}$, and $u_{3,3}$ as hyperbolic solutions and $u_{1,5}$, $u_{1,7}$, $u_{3,5}$, and $u_{3,7}$ as trigonometric solutions. To save space, the trigonometric solutions are not written in this paper, but using the hints at the end of Sec, 3.1 obtaining these solutions should not be difficult for the reader.

In graphical representations, as μ takes different values, the used ranges are limited to $-10 < x < 10$, $-10 < t < 10$ with $A = 1$ and $B = 1$. The first three plots in the figures are the 3D representation of the solutions with different μ values. The fourth graph in the figures shows the 2D representation of the solutions with different amplitudes for different μ values. Also, the change in the wave velocity c can easily be traced by considering the relation with μ.

As can be seen from Figs. 1 and 2, the maximum amplitude of the waves decreases as μ decreases in the hyperbolic type wave solutions. However, the structure of the wave becomes more predictable. Besides, in Figs. 3 and 4, the maximum amplitude of the wave increases when the value of μ increases in the trigonometric-type wave solutions. However, the structure of the wave becomes more unpredictable. Hence, the maximum amplitude of the waves and μ change proportionally.

Similarly, we can see from Figs. 5 and 6 that the maximum amplitude of the waves decreases when the value of μ decreases in the hyperbolic-type wave solutions. However, the structure of the wave becomes more predictable. On the other hand, in Figs. 7 and 8, the maximum amplitude of the waves increases when the value of μ increases in the trigonometric-type wave solutions. Also, the structure of the wave becomes more unpredictable.

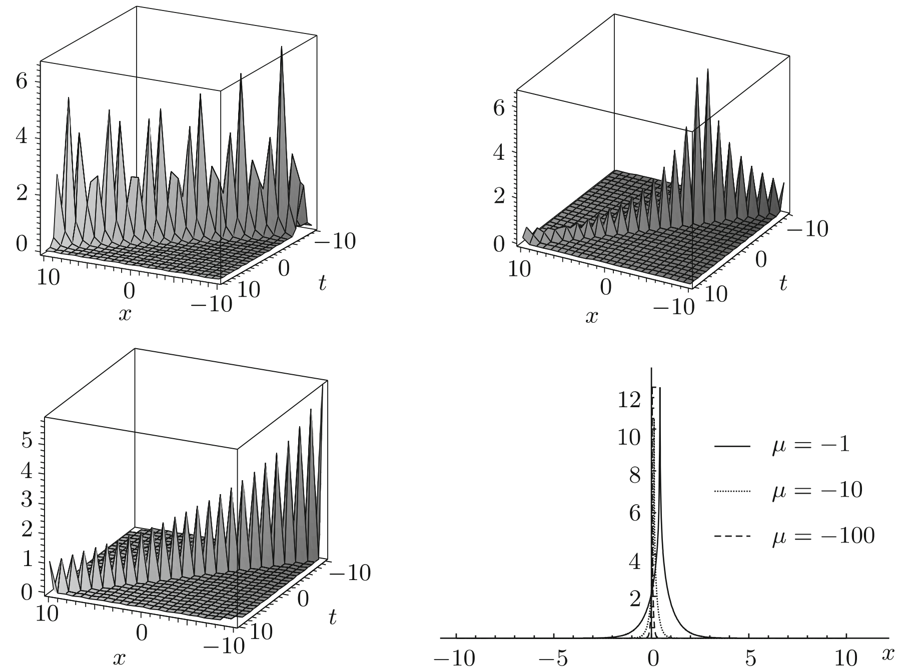

Fig. 1. The 2D and 3D plots of the solution $u_{1,1}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{-1, -10, -100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

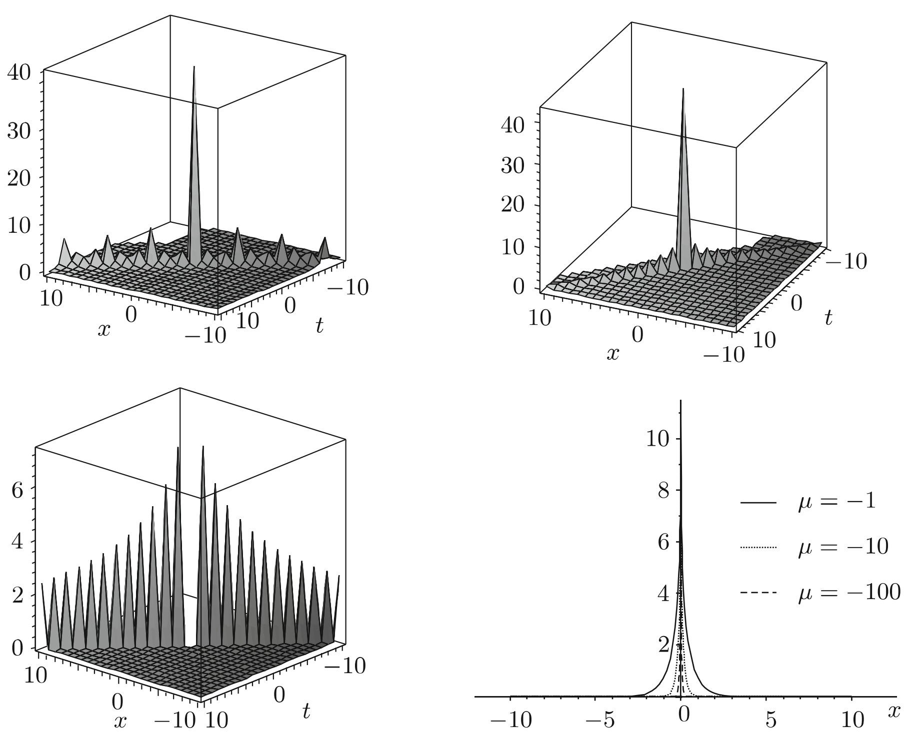

Fig. 2. The 2D and 3D plots of the solution $u_{1,3}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{-1, -10, -100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

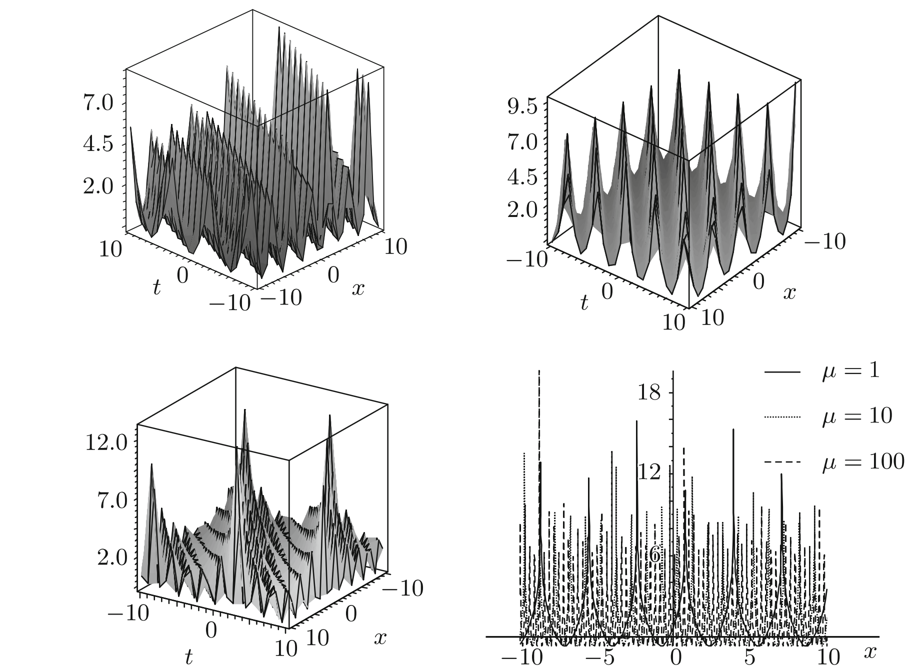

Fig. 3. The 2D and 3D plots of the solution $u_{1,5}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{1, 10, 100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

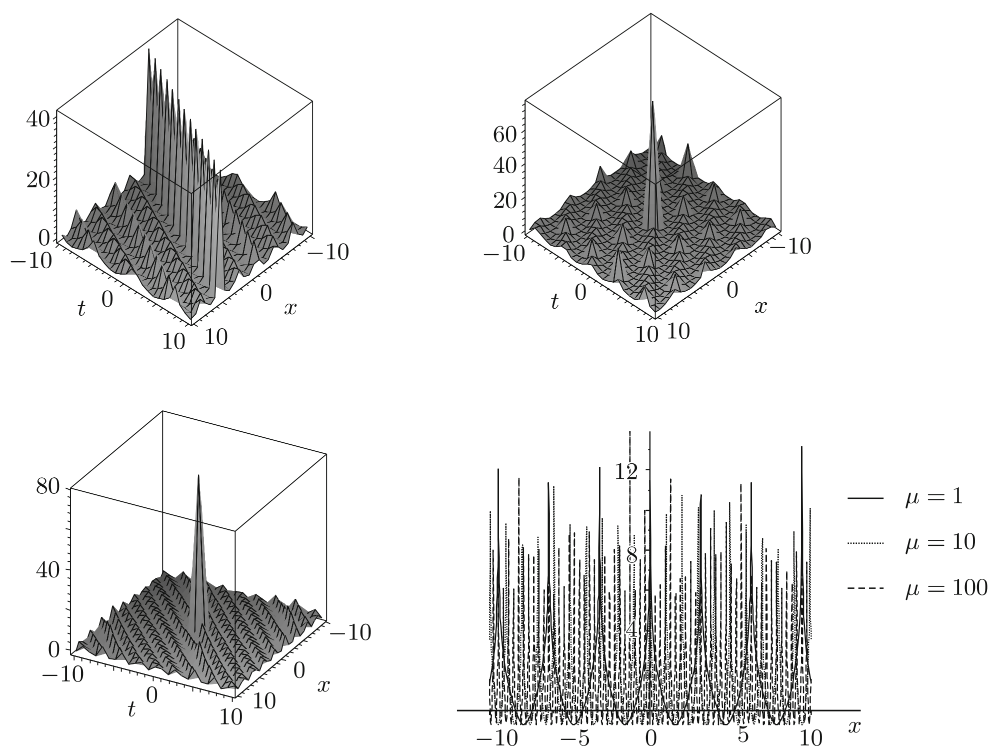

Fig. 4. The 2D and 3D plots of the solution $u_{1,7}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{1, 10, 100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

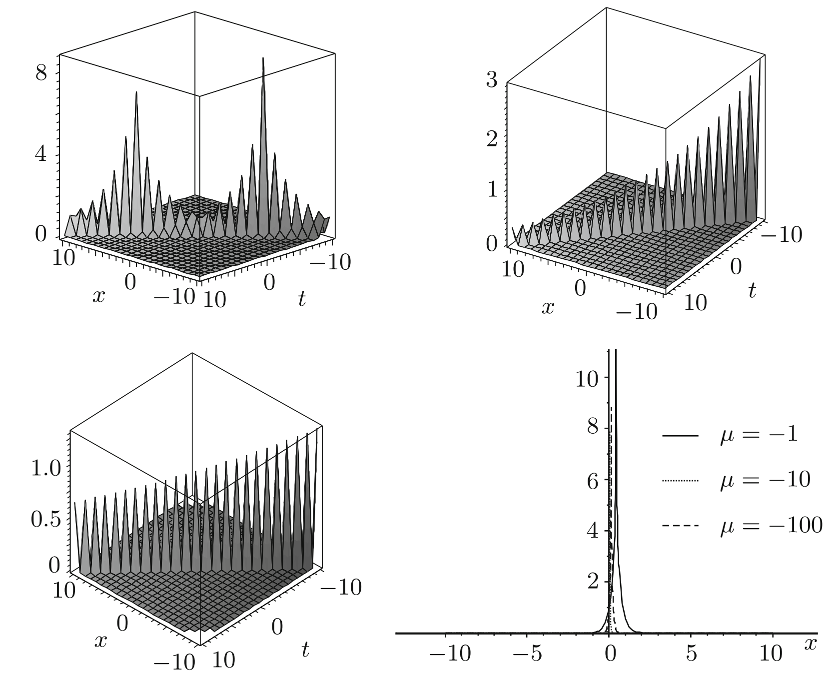

Fig. 5. The 2D and 3D plots of the solution $u_{3,1}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{-1, -10, -100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

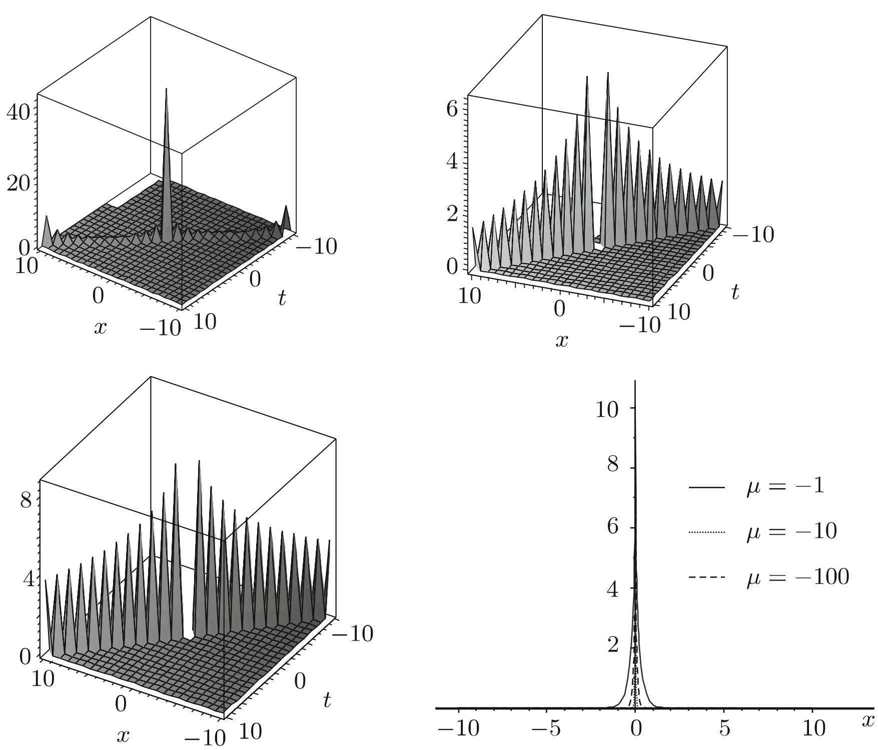

Fig. 6. The 2D and 3D plots of the solution $u_{3,3}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{-1, -10, -100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

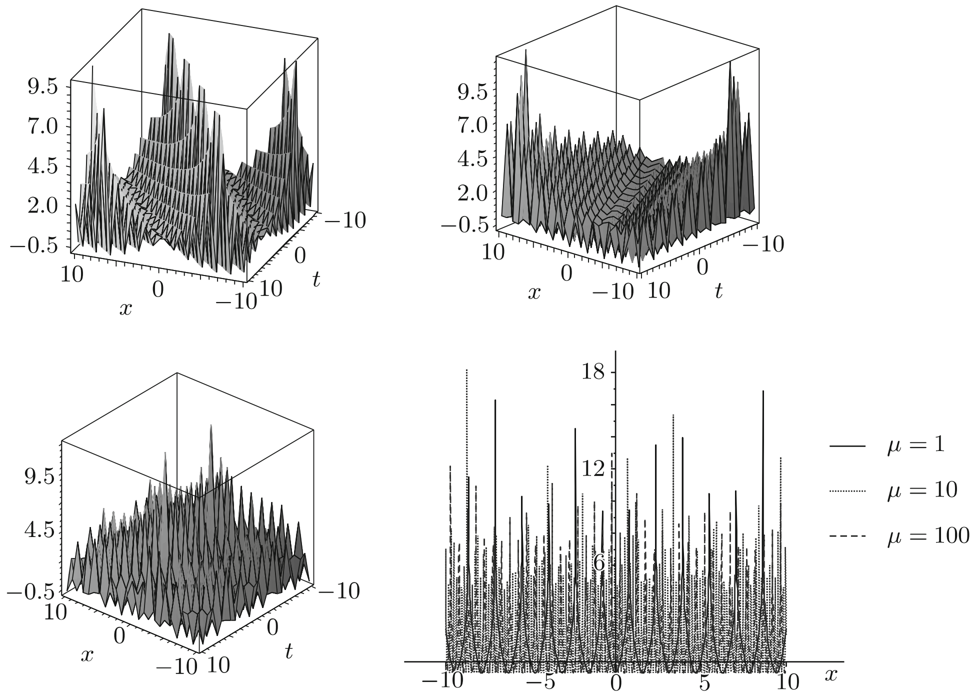

Fig. 7. The 2D and 3D plots of the solution $u_{3,5}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{1, 10, 100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

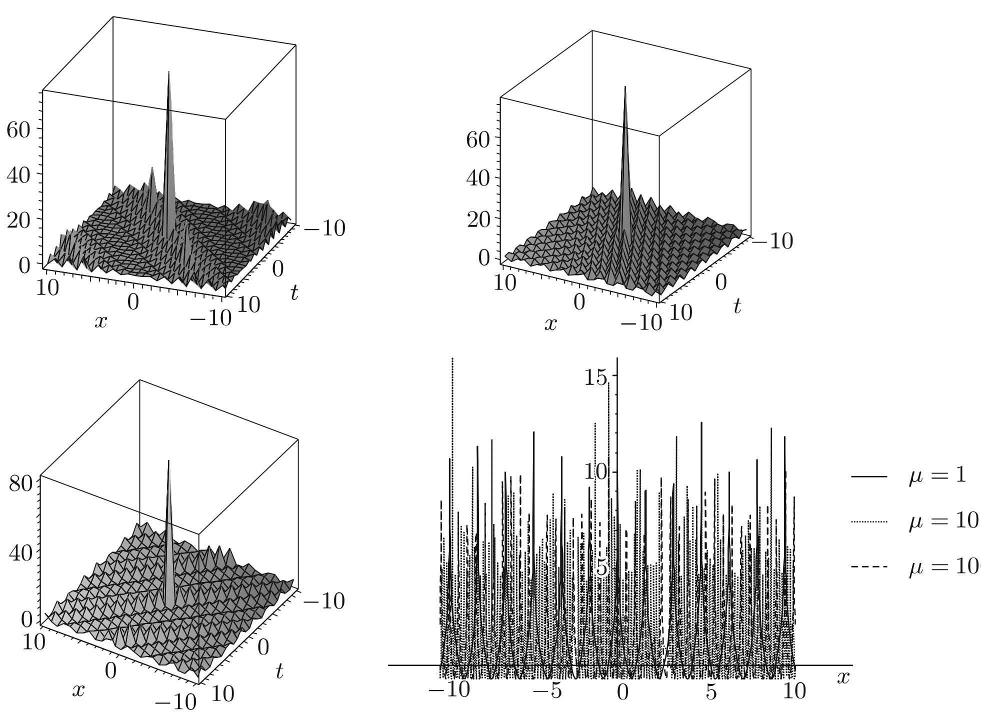

Fig. 8. The 2D and 3D plots of the solution $u_{3,7}$ with the parameters $A = 1$, $B = 1$, and $\mu = \{1, 10, 100\}$ on the intervals $-10 < x < 10$, $-10 < t < 10$.

5. Conclusions

There are many methods to solve NPDEs, but it is necessary to combine several of them to obtain many solutions. It is one of the main advantages of the unified method compared with other methods that it provides many solutions in a general way using free parameters. For example, taking $B = 0$ yields modified extended tanh method solutions, known as the strongest tanh technique. In addition, as a straightforward technique, it reduces the number of computational work. Therefore, the method is applied easily using some symbolic computation program such as `Maple` and `Mathematica`.

In this paper, the unified method is applied successfully to the Tzitzéica-type equation, which has many applications in solid state physics, nonlinear dynamics, nonlinear optics, and quantum field theory problems. Having constructed many solutions in Sec. 3, we presented the 2D and 3D plots of some selected solutions. Researchers in solid-state physics, nonlinear dynamics, nonlinear optics, and quantum field theory can use these solutions to model their own problems. Also, the differences between the structures of the solutions can be observed using various arbitrary parameters values for A, B, and μ with some other ranges of x and t.

For future studies, the author aims to improve this method to have more solutions for other NPDEs and FNPDEs.

Conflicts of interest. The author declares no conflicts of interest.

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