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Computing Periodic and Antiperiodic Eigenvalues with a PT-Symmetric Optical Potential*

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Abstract—We give estimates for the eigenvalues of nonself-adjoint Sturm–Liouville operators with periodic and antiperiodic boundary conditions for the special potential $4\cos^2 x + 4iV\sin 2x$ that is a PT-symmetric optical potential, especially when $|\sqrt{1-4V^2}| < 3$ or equally $0 \le V < \sqrt{10}/2$. We provide some useful equations for calculating the periodic and antiperiodic eigenvalues. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we give a numerical example with error analysis.

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1. INTRODUCTION AND PRELIMINARY RESULTS

In this paper, we are concerned with the operators $L_t(q)$, for $t = 0, 1$, generated in $L_2[0,\pi]$ by the differential expression

$$-y''(x) + q(x)y(x) \tag{1}$$

and the boundary conditions

$$y(\pi) = e^{i\pi t}y(0), \qquad y'(\pi) = e^{i\pi t}y'(0) \tag{2}$$

that is, periodic and antiperiodic boundary conditions, where q is a PT-symmetric optical potential of the form

$$q(x) = (1+2V)e^{i2x} + (1-2V)e^{-i2x}, \qquad V \ge 0. \tag{3}$$

which is a shift of $4\cos^2 x + 4iV\sin 2x$.

A literature review on the optical potential (3) and the general PT-symmetric potentials was given in [1] (see also [2]–[11] and references therein). In [1], we considered the Dirichlet operator $D(q)$ generated in $L_2[0,\pi]$ by expression (1) and Dirichlet boundary conditions

$$y(\pi) = y(0) = 0,$$

with the optical potential (3), proved some results concerning the Dirichlet eigenvalues and provided some useful equations for calculating Dirichlet eigenvalues. It is known that (see [12] and [5, Summary 3]) if $V \neq 1/2$, then any periodic eigenvalue is either a Dirichlet eigenvalue or a Neumann eigenvalue. Similarly, any antiperiodic eigenvalue is either a Dirichlet eigenvalue or a Neumann eigenvalue. In the present paper, we use completely different iteration formulas than those of [1] and we find completely different equations for calculating periodic and antiperiodic eigenvalues of the Schrödinger operator $L(q)$, generated in $L_2(-\infty,\infty)$ by differential expression (1) with potential (3). In the present paper, we prove the counterparts of the results in [1], mainly arguing along the same lines

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and we obtain consistent results with those of [1], by using completely different iteration formulas and equations.

The case $V = 1/2$ was investigated for the first time by Gasymov [13], and it was proved that the spectrum of the Schrödinger operator $L(q)$ is $[0, \infty)$. This case was considered also in [14], [15].

It was proved by Veliev [16, Theorem 1 and (26)] that if $ab = cd$, where a, b, c, and d are arbitrary complex numbers, then the Hill operators $L(q)$ and $L(p)$ generated in $L_2(-\infty, \infty)$ by expression (1) with the potentials $q(x) = ae^{-i2x} + be^{i2x}$ and $p(x) = ce^{-i2x} + de^{i2x}$, have the same Hill discriminant, and hence the same Bloch eigenvalues and spectrum. Therefore, the investigations of the operators $L_t(q)$, for $t = 0, 1$, can be reduced to the investigations of the operators generated in $L_2[0, \pi]$ by differential expression (1) and the boundary conditions (2) with the potential

$$p(x) = ce^{2ix} + ce^{-2ix} = 2c\cos(2x), \tag{4}$$

where $c = \sqrt{1 - 4V^2}$. Therefore, to consider the spectra of the operators $L_0(q)$ and $L_1(q)$, we can use the properties of both the PT-symmetric potential (3) and the even potential (4). The eigenvalues of $L_0(q)$ and $L_1(q)$ are called the periodic and antiperiodic eigenvalues and they are denoted by $\lambda_n(q)$, for $n \in \mathbb{Z}$ and $\mu_n(q)$, for $n \in \mathbb{Z} - \{0\}$, respectively. We may also use the notations $\lambda_n(c)$, for $n \in \mathbb{Z}$ and $\mu_n(c)$, for $n \in \mathbb{Z} - \{0\}$, for the periodic and antiperiodic eigenvalues, respectively.

In this paper, we give estimates for the periodic and antiperiodic eigenvalues, in particular, when $|c| < 3$ or correspondingly $0 \le V < \sqrt{10}/2$. We provide some useful equations for calculating the periodic eigenvalues also for the case $|c| \ge 3$ or equally $V \ge \sqrt{10}/2$. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we point out (see Remark 2) that the $(n+2)$nd critical point $V = V_{n+2}$ defined in [5], [10] as the point (see Definition 1 and Remark 1) at which the $(2n+1)$th and $(2n+2)$th periodic eigenvalues are equal and real satisfies the inequality

$$V_{n+2} \ge \frac{\sqrt{(2n+1)^2 + 1}}{2}$$

for $n = 1, 2, \dots$ under the assumption that the conjectures about the critical points given by Veliev [6] are valid. Obviously, $\lim_{n\to\infty} V_n = \infty$ as stated by Veliev [6, Conjecture 3]. Finally, we give a numerical example for $c^2 = -2.157281295$ with error analysis using Rouche's theorem.

For simplicity of reading, first, we give the main ideas of the proofs of the main results. We will focus on the periodic eigenvalues. The investigation of the antiperiodic eigenvalues is similar. To give estimates for the small periodic eigenvalues, first, we prove (See Theorem 1) that the periodic eigenvalues satisfy the equation

$$\left(\lambda - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda)\right)^2 = \left(\frac{q_{-1}}{q_1}\right)^{2n} \left(\sum_{k=1}^{\infty} B_{2k-1}(\lambda)\right)^2,$$

for $|c| \le 3$ and $n \ge 2$, where $q_{-1} = 1 - 2V$, $q_1 = 1 + 2V$ and the infinite series A_k and B_k are defined in (11). This implies that the periodic eigenvalue $\lambda_{\pm n}$ is either the root of (14) or the root of (15) lying in the disk $D_n = \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \le 2|c|\}$. Then we consider the first periodic eigenvalues λ_0 and λ_{-1} and the third periodic eigenvalue λ_{+1} in Theorem 2 and prove that for $|c| < 3$ the eigenvalues λ_0 and λ_{-1} are roots of Eq. (20) in the disk $D_1 = \{\lambda \in \mathbb{C}\colon |\lambda| \le 2|c| + 4\}$ and that λ_{+1} is a root of (21) in the disk D_1. Then, to estimate the eigenvalues numerically, we take finite sums instead of the infinite series in Eqs. (14), (15), (20), and (21) and approximate the eigenvalues by the roots of the polynomials derived from the mth approximations (24), (25), and (26) the way it was done by Veliev [5]. We obtain similar results for the antiperiodic eigenvalues in Theorems 3 and 4.

Now we state some preliminary facts. It is well known that the spectra of the operators $L_0(q)$ and $L_1(q)$ are discrete and for sufficiently large n there exist two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counting multiplicities) in a neighborhood of n^2. See the basic and detailed classical results in [17]–[21] and references therein. The eigenvalues of the operators $L_0(0)$ and $L_1(0)$ are $(2n)^2$ and $(2n+1)^2$, for $n \in \mathbb{Z}$, respectively and all eigenvalues of $L_0(0)$ and $L_1(0)$, except 0, are double.

It is also known [19], [21] that if c is a real nonzero number, then all eigenvalues of the operator $H_t(c)$, generated in $L_2[0,\pi]$ by expression (1) and the boundary conditions (2) with potential (4), are real and simple. These results were stated more precisely in [5, Summary 2].

By [6, Theorem 9], for complex values of c, the eigenvalues of the operator $H_0(c)$ lie in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \le 2|c|\}$, for $n = 0, 1, 2, \dots$ and $|c| < 3$. Moreover, the disk D_n, for $n \ge 2$, has no common points with another disk D_m, for $m \ne n$ and the boundary of the disk $D_{n,\epsilon} := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \le 2|c| + \epsilon\}$, $n = 2, 3 \dots$, belongs to the resolvent set of the operator $H_0(c)$ for all $|c| < 3$ if ϵ is a sufficiently small positive number. It implies that the number of eigenvalues (counting multiplicities) of $H_0(c)$ in $D_{n,\epsilon}$ for $n \ge 2$ is the same for all $|c| < 3$. Since $H_0(0)$ has two eigenvalues in $D_{n,\epsilon}$, for $n \ge 2$, the operator $H_0(c)$ has also two eigenvalues for $|c| < 3$. Letting ϵ tend to zero, we obtain that $H_0(c)$ has two eigenvalues (counting the multiplicity) in D_n, for $n \ge 2$ and $|c| < 3$. By the same token, we prove that $H_0(c)$ has 3 eigenvalues in $D_0 \cup D_1$. We denote them by λ_0, λ_{-1}, and λ_{+1}.

Similarly, $H_1(c)$ has two eigenvalues (counting multiplicities) in

$$d_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| \le 2|c|\}$$

for $n = 1, 2, \dots$ and $|c| < 2$. We denote the $(2n)$th and $(2n+1)$th periodic eigenvalues by $\lambda_{-n}(c)$ and $\lambda_{+n}(c)$, for $n = 1, 2, \dots$; the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues by $\mu_{-n}(c)$ and $\mu_{+n}(c)$, for $n = 1, 2, \dots$, respectively.

Thus,

$$|\lambda_{\pm n}(c) - \lambda_{\pm n}(0)| \le 2|c|,$$

for $n \ge 2$, $|c| < 3$, and

$$|\mu_{\pm n}(q) - \mu_{\pm n}(0)| \le 2|c|,$$

for $n \ge 1$ and $|c| < 2$, where $\lambda_{\pm n}(0) = (2n)^2$, $\mu_{\pm n}(0) = (2n-1)^2$ and $c = \sqrt{1 - 4V^2}$. Therefore, we have

$$(2n)^2 - 2|c| \le |\lambda_n| \le (2n)^2 + 2|c| \tag{5}$$

and

$$\begin{aligned} |\lambda_n - (2k)^2| &\ge |(2n)^2 - (2k)^2| - 2|c| = 4|n-k||n+k| - 2|c| \\ &\ge 4|2n-1| - 2|c|, \end{aligned}$$

for $n \ge 2$ and $k \ne \pm n$. In particular, if $n = 1$, we have $|\lambda_{\pm 1}| \le 4 + 2|c| < 10$ and

$$|\lambda_{\pm 1} - (2k)^2| \ge ||\lambda_{\pm 1}| - (2k)^2| \ge 16 - |\lambda_{\pm 1}| \ge 12 - 2|c| > 6, \tag{6}$$

for $k \ge 2$. Besides, if $n \ge 2$, we have $|\lambda_n| \ge |\lambda_{-2}| \ge 16 - 2|c| > 10$ and

$$|\lambda_n - (2k)^2| \ge ||\lambda_{-2}| - (2k)^2| \ge |\lambda_{-2}| - 4 \ge 12 - 2|c| > 6, \tag{7}$$

for $k \ne \pm n$. The analogous inequalities can be written for the antiperiodic eigenvalues from

$$(2n-1)^2 - 2|c| \le |\mu_{\pm n}| \le (2n-1)^2 + 2|c|, \tag{8}$$

for $n = 1, 2, \dots$.

2. MAIN RESULTS

First, we consider the operator $L_0(q)$ which is associated with the periodic boundary conditions. From now on, when we use the notation λ_n, we mean the $(2n)$th and $(2n+1)$th periodic eigenvalues λ_{-n} and λ_{+n}, for $n = 1, 2, \dots$. We begin with the equations

$$(\lambda_N - (2n)^2)(\Psi_N, e^{i2nx}) = (q\Psi_N, e^{i2nx}), \tag{9}$$

$$(\lambda_N - (2n)^2)(\Psi_N, e^{-i2nx}) = (q\Psi_N, e^{-i2nx}) \tag{10}$$

which are obtained from

$$-\Psi_N''(x) + q(x)\Psi_N(x) = \lambda_N \Psi_N(x),$$

by multiplying both sides of the equality by e^{i2nx} and e^{-i2nx}, respectively, where $\Psi_N(x)$ is the eigenfunction corresponding to the eigenvalue λ_N. Iterating equation (9) m times for $N = n$, as it was done in the paper [22], we obtain

$$\Big(\lambda_n - (2n)^2 - \sum_{k=1}^{m} A_k(\lambda_n)\Big)(\Psi_n, e^{i2nx}) - \sum_{k=1}^{m} B_k(\lambda_n)(\Psi_n, e^{-i2nx}) = R_m(\lambda_n), \tag{11}$$

where

$$A_k(\lambda_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k} q_{-n_1-n_2-\cdots-n_k}}{[\lambda_n - (2(n-n_1))^2]\cdots[\lambda_n - (2(n-n_1-\cdots-n_k))^2]},$$

$$B_k(\lambda_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k} q_{2n-n_1-n_2-\cdots-n_k}}{[\lambda_n - (2(n-n_1))^2]\cdots[\lambda_n - (2(n-n_1-\cdots-n_k))^2]},$$

$$R_m(\lambda_n) = \sum_{n_1,n_2,\dots,n_{m+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_m} q_{n_{m+1}}(q\Psi_n, e^{i2(n-n_1-\cdots-n_{m+1})x})}{[\lambda_n - (2(n-n_1))^2]\cdots[\lambda_n - (2(n-n_1-\cdots-n_{m+1}))^2]}.$$

Here the sums are taken under the conditions $n_s = \pm 1$, $\sum_{j=1}^{s} n_j \neq 0, 2n$ for $s = 1, 2, \dots, m+1$. Note that for the optical potential of the form (3) we have $q_{-1} = 1 - 2V$, $q_1 = 1 + 2V$ and $q_k = 0$ for $k \neq \pm 1$.

Similarly, iterating equation (10) m times, we obtain

$$\Big(\lambda_n - (2n)^2 - \sum_{k=1}^{m} A_k^*(\lambda_n)\Big)(\Psi_n, e^{-i2nx}) - \sum_{k=1}^{m} B_k^*(\lambda_n)(\Psi_n, e^{i2nx}) = R_m^*(\lambda_n), \tag{12}$$

where

$$A_k^*(\lambda_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k} q_{-n_1-n_2-\cdots-n_k}}{[\lambda_n - (2(n+n_1))^2]\cdots[\lambda_n - (2(n+n_1+\cdots+n_k))^2]},$$

$$B_k^*(\lambda_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k} q_{-2n-n_1-n_2-\cdots-n_k}}{[\lambda_n - (2(n+n_1))^2]\cdots[\lambda_n - (2(n+n_1+\cdots+n_k))^2]},$$

$$R_m^*(\lambda_n) = \sum_{n_1,n_2,\dots,n_{m+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_m} q_{n_{m+1}}(q\Psi_n, e^{-i2(n+n_1+\cdots+n_{m+1})x})}{[\lambda_n - (2(n+n_1))^2]\cdots[\lambda_n - (2(n+n_1+\cdots+n_{m+1}))^2]}.$$

Here, the sums are taken under the conditions $n_s = \pm 1$, $\sum_{j=1}^{s} n_j \neq 0, -2n$ for $s = 1, 2, \dots, m+1$. Note that the iteration formulas (11) and (12) were used in [22] for large eigenvalues to obtain asymptotic formulas. In this paper, we find conditions on potential (3) for which the iteration formulas (11) and (12) are also valid for the small eigenvalues, as m tends to infinity. We also note that it is not easy to give such conditions; there are many technical calculations. Since the potential q is the optical potential of the form (3), we have the followings, after some calculations (see [23]):

$$A_{2k}^*(\lambda_n) = A_{2k}(\lambda_n) = B_{2k}^*(\lambda_n) = B_{2k}(\lambda_n) = 0,$$
$$A_{2k-1}^*(\lambda_n) = A_{2k-1}(\lambda_n), \qquad B_{2k-1}^*(\lambda_n) = \Big(\frac{q_{-1}}{q_1}\Big)^{2n} B_{2k-1}(\lambda_n), \tag{13}$$

for $k = 1, 2, \dots$. Now, to give the main results, we prove the following lemma. Without loss of generality, we assume that $\Psi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue λ_n.

Lemma 1. *The statements*

(a) $\lim_{m\to\infty} R_m(\lambda_n) = 0, \qquad \lim_{m\to\infty} R_m^*(\lambda_n) = 0,$

(b) $|u_n|^2 + |v_n|^2 > 0$, *where* $u_n = (\Psi_n, e^{i2nx})$ *and* $v_n = (\Psi_n, e^{-i2nx})$

are valid in the following cases:

Case 1. *If* $|c| < 3$, *for all* $n \geq 1$.

Case 2. *If* $|c| \leq 2s - 1$, *for* $n \geq s$ *and* $s = 2, 3, \ldots$.

Proof. Case 1. (a) By the definition of $R_m(\lambda_n)$ and the conditions imposed on the summations, the number of summands of $R_{2m+1}(\lambda_n)$ is not greater than 4^{2m}. On the other hand, by (5)–(7), we have

$$|\lambda_1| \leq 4 + 2|c| < 10, \qquad |\lambda_1 - 16| \geq 16 - |\lambda_1| > 6, \qquad |\lambda_1 - 36| \geq 36 - |\lambda_1| > 26,$$
$$10 < 16 - 2|c| \leq |\lambda_2| \leq 16 + 2|c| < 22, \qquad |\lambda_2 - 4| \geq |\lambda_2| - 4 > 6,$$
$$|\lambda_2 - 36| \geq 36 - |\lambda_2| > 14, \qquad |\lambda_2 - 64| \geq 64 - |\lambda_2| > 42,$$
$$30 < 36 - 2|c| \leq |\lambda_3| \leq 36 + 2|c| < 42,$$
$$|\lambda_3 - 16| \geq |\lambda_3| - 16 > 14, \qquad |\lambda_3 - 64| \geq 64 - |\lambda_3| > 22.$$

Hence, considering the greatest summands of $R_{2m+1}(\lambda_n)$ in absolute value, we obtain for $n = 1$,

$$|R_{2m+1}(\lambda_1)| < \frac{4^{2m}|q_{-1}|^{m+1}|q_1|^m 2|c|\sqrt{\pi}}{|\lambda_1 - 16|^{m+1}|\lambda_1 - 36|^m} < \frac{2\sqrt{\pi}|q_{-1}|4^{2m}|c|^{2m+1}}{6^{m+1}26^m}$$
$$< \frac{\sqrt{\pi}|q_{-1}|12^{2m}}{6^m 26^m} = \frac{\sqrt{\pi}|q_{-1}|24^m}{26^m} = \sqrt{\pi}|q_{-1}|\left(\frac{12}{13}\right)^m,$$

for $n = 2$,

$$|R_{2m+1}(\lambda_2)| < \frac{4^m|q_{-1}|^m|q_1|^{m+1}2|c|\sqrt{\pi}}{|\lambda_2 - 4|^{m+1}|\lambda_2|^m} + \frac{4^{2m}|q_{-1}|^{m+1}|q_1|^m 2|c|\sqrt{\pi}}{|\lambda_2 - 36|^{m+1}|\lambda_2 - 64|^m}$$
$$< \frac{2\sqrt{\pi}|q_1|4^m|c|^{2m+1}}{6^{m+1}10^m} + \frac{2\sqrt{\pi}|q_{-1}|4^{2m}|c|^{2m+1}}{14^{m+1}42^m}$$
$$< \frac{\sqrt{\pi}|q_{-1}|12^m}{2^m 10^m} + \frac{6\sqrt{\pi}|q_{-1}|12^m 12^m}{14.14^m 42^m} = \sqrt{\pi}|q_{-1}|\left(\frac{3}{5}\right)^m + \frac{3\sqrt{\pi}|q_{-1}|}{7}\left(\frac{36}{147}\right)^m,$$

and in general, for $n \geq 3$ we have $|R_{2m+1}(\lambda_n)| < ar^m$, for some constant $a > 0$ and $0 < r < 1$. Therefore, $\lim_{m\to\infty} R_m(\lambda_n) = 0$. Similarly, we prove that $\lim_{m\to\infty} R_m^*(\lambda_n) = 0$.

(b) Suppose the contrary, $u_n = 0$ and $v_n = 0$. Since the system of root functions $\{e^{2ikx}/\sqrt{\pi} : k \in \mathbb{Z}\}$ of $L_0(0)$ forms an orthonormal basis for $L_2[0, \pi]$, we have the decomposition

$$\pi\Psi_n = u_n e^{i2nx} + v_n e^{-i2nx} + \sum_{k\in\mathbb{Z}, k\neq\pm n} \left(\Psi_n, e^{i2kx}\right) e^{i2kx}$$

for the normalized eigenfunction Ψ_n corresponding to the eigenvalue λ_n of $L_0(q)$. By Parseval's equality, we obtain

$$\sum_{k\in\mathbb{Z}, k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi.$$

First, we consider the case $n = 1$. Using the relations (6) and (9), the Bessel inequality, and taking $(q\Psi_1, 1) = q_{-1}u_1 + q_1 v_1 = 0$ into account, we obtain

$$\sum_{k\in\mathbb{Z}, k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 = \frac{|(q\Psi_1, 1)|^2}{|\lambda_1|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_1, e^{i2kx})|^2}{|\lambda_1 - (2k)^2|^2}$$
$$\leq \frac{1}{(12 - 2|c|)^2} \sum_{k\in\mathbb{Z}, k\neq\pm 1} |(q\Psi_1, e^{i2kx})|^2 < \frac{\pi(2|c|)^2}{6^2} < \pi,$$

and in the case $n \geq 2$, we have

$$\sum_{k\in\mathbb{Z},k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \sum_{k\in\mathbb{Z},k\neq\pm n} \frac{|(q\Psi_n, e^{i2kx})|^2}{|\lambda_n - (2k)^2|^2}$$
$$\leq \frac{1}{(12-2|c|)^2} \sum_{k\in\mathbb{Z},k\neq\pm n} |(q\Psi_n, e^{i2kx})|^2 < \frac{\pi(2|c|)^2}{6^2} < \pi,$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi$ and completes the proof for Case 1.

Case 2. First, we prove the statements for $n = 2$ and $|c| = 3$. The proof of (a) is obvious. For the proof of (b), again assume the contrary $u_2 = (\Psi_2, e^{i4x}) = 0$ and $v_2 = (\Psi_2, e^{-i4x}) = 0$. Isolating the terms $|(\Psi_2, e^{i2x})|^2$ and $|(\Psi_2, e^{-i2x})|^2$ in Parseval's equality, we can write

$$|(\Psi_2, e^{-i2x})|^2 + |(\Psi_2, e^{i2x})|^2 + \sum_{k\neq\pm1,\pm2} |(\Psi_2, e^{i2kx})|^2 = \pi.$$

Using (9), the relations $|\lambda_2| \geq 16 - 2|c| = 10$ and $|\lambda_2 - 4| \geq |\lambda_2| - 4 \geq 12 - 2|c| = 6$, the Bessel inequality, and taking

$$(q\Psi_2, e^{-i2x}) = q_{-1}(\Psi_2, 1) + q_1 v_2 = \frac{q_{-1}(q\Psi_2, 1)}{\lambda_2},$$
$$(q\Psi_2, e^{i2x}) = q_{-1}u_2 + q_1(\Psi_2, 1) = \frac{q_1(q\Psi_2, 1)}{\lambda_2}$$

into account, we obtain

$$\sum_{k\in\mathbb{Z},k\neq\pm2} |(\Psi_2, e^{i2kx})|^2 = \frac{|(q\Psi_2, e^{-i2x})|^2}{|\lambda_2 - 4|^2} + \frac{|(q\Psi_2, e^{i2x})|^2}{|\lambda_2 - 4|^2} + \sum_{k\neq\pm1,\pm2} \frac{|(q\Psi_2, e^{i2kx})|^2}{|\lambda_2 - (2k)^2|^2}$$
$$\leq \frac{(1-\sqrt{10})^2|(q\Psi_2, 1)|^2}{|\lambda_2|^2|\lambda_2 - 4|^2} + \frac{(1+\sqrt{10})^2|(q\Psi_2, 1)|^2}{|\lambda_2|^2|\lambda_2 - 4|^2} + \sum_{k\neq\pm1,\pm2} \frac{|(q\Psi_2, e^{i2kx})|^2}{|\lambda_2|^2}$$
$$\leq \frac{(1-\sqrt{10})^2\pi(2|c|)^2}{10^2 6^2} + \frac{(1+\sqrt{10})^2\pi(2|c|)^2}{10^2 6^2} + \frac{1}{10^2} \sum_{k\in\mathbb{Z},k\neq\pm2} |(q\Psi_2, e^{i2kx})|^2$$
$$\leq \frac{\pi(1-\sqrt{10})^2}{10^2} + \frac{\pi(1+\sqrt{10})^2}{10^2} + \frac{\pi(2|c|)^2}{10^2} = \frac{58\pi}{100} < \pi,$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq\pm2} |(\Psi_2, e^{i2kx})|^2 = \pi$ and completes the proof for the case $n = 2$ and $|c| = 3$.

Now consider the case of $|c| \leq 2s - 1$ and $n \geq s$ for $s \geq 3$. Using $(2n)^2 - 2|c| \leq |\lambda_n| \leq (2n)^2 + 2|c|$, we obtain

$$|\lambda_n - (2k)^2| \geq |\lambda_n - (2(n-1))^2| \geq (2n)^2 - 2|c| - (2(n-1))^2$$
$$= 4(2n-1) - 2|c| \geq 4(2s-1) - 2(2s-1) = 4s - 2,$$

and for $k \neq n - 1$, we have

$$|\lambda_n - (2k)^2| \geq |\lambda_n - (2(n+1))^2| \geq (2(n+1))^2 - (2n)^2 - 2|c|$$
$$= 4(2n+1) - 2|c| \geq 4(2s+1) - 2(2s-1) = 4s + 6.$$

Arguing the case, $n = s$ and $|c| = 2s - 1$, as in the case, $n = 2$ and $|c| = 3$, and repeating the proof of Lemma 1 by using the last inequalities, we complete the proof. $\square$

Now, letting m tend to infinity in the equations (11) and (12), we obtain the following results. First, we consider the case $n \geq 2$.

Theorem 1. **(a)** *If $|c| \le 3$ and $n \ge 2$, then $\lambda_{\pm n}$ is an eigenvalue of $L_0(q)$ if and only if it is the root of either the equation*

$$\lambda - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda) - \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{\infty} B_{2k-1}(\lambda) = 0 \tag{14}$$

or the equation

$$\lambda - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda) + \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{\infty} B_{2k-1}(\lambda) = 0 \tag{15}$$

in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \le 2|c|\}$, where $q_{-1} = 1 - 2V$, $q_1 = 1 + 2V$, and each of the series in these equations converges uniformly to an analytic function on the disk D_n. Moreover, the roots of (14) and (15) in D_n coincide with the $2n$th and $(2n+1)$st periodic eigenvalues λ_{-n} and λ_{+n} of L_0.

(b) *If $|c| \le 2s - 1$, $s = 2, 3, \dots$, then the statements in* **(a)** *remain valid for $n \ge s$.*

Proof. **(a)** By Lemma 1, letting m tend to infinity in Eqs. (11) and (12), we obtain

$$\left(\lambda_n - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda_n)\right) u_n = \sum_{k=1}^{\infty} B_{2k-1}(\lambda_n) v_n, \tag{16}$$

$$\left(\lambda_n - (2n)^2 - \sum_{k=1}^{\infty} A^*_{2k-1}(\lambda_n)\right) v_n = \sum_{k=1}^{\infty} B^*_{2k-1}(\lambda_n) u_n, \tag{17}$$

where $u_n = (\Psi_n, e^{i2nx})$ and $v_n = (\Psi_n, e^{-i2nx})$. If one of the numbers u_n and v_n is zero, then the proof is obvious. If they are both different from zero, multiplying these equations side by side and then cancelling the term $u_n v_n$, by (13), we obtain

$$\left(\lambda_n - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda_n)\right)^2 = \left(\frac{q_{-1}}{q_1}\right)^{2n} \left(\sum_{k=1}^{\infty} B_{2k-1}(\lambda_n)\right)^2, \tag{18}$$

which implies λ_n is either the root of (14) or the root of (15), since $B_1(\lambda_n) = 0$ for $n \ge 2$.

Now we prove that the roots of (14) and (15) lying in the disk D_n are the eigenvalues of L_0. The equation $f(\lambda) := \lambda - (2n)^2 - A_1(\lambda) = 0$ has one root in the disk D_n and

$$\begin{aligned} |f(\lambda)| &= |\lambda - (2n)^2 - A_1(\lambda)| \ge \big||\lambda - (2n)^2| - |A_1(\lambda_2)|\big| \\ &\ge \left| |\lambda - (2n)^2| - \left(\frac{|c|^2}{|\lambda_2 - 4|} + \frac{|c|^2}{|\lambda_2 - 36|}\right)\right| \ge 2|c| - \left(\frac{|c|^2}{12 - 2|c|} + \frac{|c|^2}{20 - 2|c|}\right), \end{aligned}$$

for all $\lambda \in C_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| = 2|c|\}$. Define

$$g_j(\lambda) := \lambda - (2n)^2 - \sum_{k=1}^{\infty} A_{2k-1}(\lambda) + (-1)^j \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{\infty} B_{2k-1}(\lambda),$$

for $j = 1, 2$. Estimating the summands of $|A_{2k-1}(\lambda_2)|$ and $|B_{2k-1}(\lambda_2)|$, we obtain

$$|A_{2k-1}(\lambda_2)| < \frac{2^{k-1}|c|^{2k}}{|\lambda_2|^{k-1}|\lambda_2 - 4|^k}, \qquad \left|\frac{q_{-1}}{q_1}\right|^2 |B_{2k-1}(\lambda_2)| < \frac{2^{k-2}|c|^{2k}}{|\lambda_2|^{k-1}|\lambda_2 - 4|^k},$$

for $k \ge 2$. Using the relations $|\lambda_2| \ge 16 - 2|c|$ and $|\lambda_2 - 4| \ge |\lambda_2| - 4 \ge 12 - 2|c|$, it follows by the geometric series formula that

$$\sum_{k=2}^{\infty} |A_{2k-1}(\lambda_2)| < \frac{2|c|^4}{(16 - 2|c|)(12 - 2|c|)^2} + \frac{2^2|c|^6}{(16 - 2|c|)^2(12 - 2|c|)^3} + \cdots$$

$$= \frac{2|c|^4}{(16-2|c|)(12-2|c|)^2}(1+\frac{2|c|^2}{(16-2|c|)(12-2|c|)}+\frac{2^2|c|^4}{(16-2|c|)^2(12-2|c|)^2}+\cdots)$$
$$= \frac{2|c|^4}{(16-2|c|)(12-2|c|)^2}\frac{1}{1-\frac{2|c|^2}{(16-2|c|)(12-2|c|)}} = \frac{2|c|^4}{(12-2|c|)[(16-2|c|)(12-2|c|)-2|c|^2]} < \frac{9}{14}$$

and that

$$\left|\frac{q_{-1}}{q_1}\right|^2\sum_{k=2}^{\infty}|B_{2k-1}(\lambda_2)| < \frac{|c|^4}{(16-2|c|)(12-2|c|)^2}+\frac{2|c|^6}{(16-2|c|)^2(12-2|c|)^3}+\cdots$$
$$= \frac{|c|^4}{(16-2|c|)(12-2|c|)^2}(1+\frac{2|c|^2}{(16-2|c|)(12-2|c|)}+\frac{2^2|c|^4}{(16-2|c|)^2(12-2|c|)^2}+\cdots)$$
$$= \frac{|c|^4}{(16-2|c|)(12-2|c|)^2}\frac{1}{1-\frac{2|c|^2}{(16-2|c|)(12-2|c|)}} = \frac{|c|^4}{(12-2|c|)[(16-2|c|)(12-2|c|)-2|c|^2]}$$
$$< \frac{9}{28}.$$

Hence

$$|g_j(\lambda)-f(\lambda)| = |\sum_{k=2}^{\infty}A_{2k-1}(\lambda) \pm \left(\frac{q_{-1}}{q_1}\right)^n\sum_{k=2}^{\infty}B_{2k-1}(\lambda)|$$
$$\le \sum_{k=2}^{\infty}|A_{2k-1}(\lambda)| + \left|\frac{q_{-1}}{q_1}\right|^n\sum_{k=2}^{\infty}|B_{2k-1}(\lambda)|$$
$$\le \sum_{k=2}^{\infty}|A_{2k-1}(\lambda_2)| + \left|\frac{q_{-1}}{q_1}\right|^2\sum_{k=2}^{\infty}|B_{2k-1}(\lambda_2)| < \frac{9}{14}+\frac{9}{28}=\frac{27}{28}.$$

for all $\lambda \in C_n$. Therefore $|g_j(\lambda)-f(\lambda)| < |f(\lambda)|$ holds for all $\lambda \in C_n$ and by Rouche's theorem, $g_j(\lambda)$ has one root in the disk D_n, for $j=1$ and $j=2$. Hence, L_0 has one eigenvalue (counting with multiplicity) lying in D_n, which is the root of (14) and it has one eigenvalue (counting with multiplicity) lying in D_n, which is the root of (15). On the other hand, each of the equations (14) and (15) has exactly one root (counting with multiplicity) in D_n. Thus, $\lambda \in D_n$ is an eigenvalue of L_0 if and only if, it is either the root of (14) or the root of (15) and the roots of (14) and (15) coincide with the eigenvalues λ_{-n} and λ_{+n} of L_0.

Now, to estimate $\sum_{k=2}^{\infty}|A'_{2k-1}(\lambda)|$ and $\sum_{k=2}^{\infty}|B'_{2k-1}(\lambda)|$, for $|\lambda-(2n)^2| \le 2|c|$ and $|c| \le 3$, we first estimate the summands $|A'_{2k-1}(\lambda_2)|$ and $|\frac{q_{-1}}{q_1}|^2|B'_{2k-1}(\lambda_2)|$ by differentiating $A_{2k-1}(\lambda_2)$ and $B_{2k-1}(\lambda_2)$ with respect to λ_2:

$$|A'_{2k-1}(\lambda_2)| < \frac{3(\frac{3}{2})^{k-2}|c|^{2k}}{|\lambda_2|^{k-1}|\lambda_2-4|^{k+1}}, \qquad \left|\frac{q_{-1}}{q_1}\right|^2|B'_{2k-1}(\lambda_2)| < \frac{3(\frac{5}{2})^{k-2}|c|^{2k}}{|\lambda_2|^{k-1}|\lambda_2-4|^{k+1}},$$

for $k \ge 2$, and hence,

$$\sum_{k=2}^{\infty}|A'_{2k-1}(\lambda)| \le \sum_{k=2}^{\infty}|A'_{2k-1}(\lambda_2)| < \frac{3|c|^4}{(16-2|c|)(12-2|c|)^3}+\frac{(9/2)|c|^6}{(16-2|c|)^2(12-2|c|)^4}+\cdots$$
$$= \frac{3|c|^4}{(16-2|c|)(12-2|c|)^3}(1+\frac{(3/2)|c|^2}{(16-2|c|)(12-2|c|)}+\frac{(9/4)|c|^4}{(16-2|c|)^2(12-2|c|)^2}\cdots)$$
$$= \frac{3|c|^4}{(16-2|c|)(12-2|c|)^3}\frac{1}{1-\frac{3|c|^2}{2(16-2|c|)(12-2|c|)}} = \frac{6|c|^4}{(12-2|c|)^2[2(16-2|c|)(12-2|c|)-3|c|^2]}$$
$$< \frac{9}{62}$$

and

$$\left|\frac{q_{-1}}{q_1}\right|^n \sum_{k=2}^{\infty} |B'_{2k-1}(\lambda)| \leq \left|\frac{q_{-1}}{q_1}\right|^2 \sum_{k=2}^{\infty} |B'_{2k-1}(\lambda_2)| < \frac{6|c|^4}{(12-2|c|)^2[2(16-2|c|)(12-2|c|)-5|c|^2]} < \frac{9}{50}.$$

Therefore, each of the series $\sum_{k=1}^{\infty} A_{2k-1}(\lambda)$ and $\sum_{k=1}^{\infty} B_{2k-1}(\lambda)$ converges uniformly to an analytic function on the disk D_n.

(b) This is obvious by Lemma 1 (Case2). □

Now consider the case of $n = 1$. In this case, calculating $A_1(\lambda_1)$ and $\sum_{k=1}^{\infty} B_{2k-1}(\lambda_1)$ and substituting them in (11) and (12) as $m \to \infty$, we obtain

$$\left(\lambda_1 - 4 - \frac{c^2}{\lambda_1} - \frac{c^2}{\lambda_1 - 16} - \sum_{k=2}^{\infty} A_{2k-1}(\lambda_1)\right)^2 = \frac{c^4}{\lambda_1^2},$$

or

$$\left(\lambda_1 - 4 - \frac{2c^2}{\lambda_1} - \frac{c^2}{\lambda_1 - 16} - \sum_{k=2}^{\infty} A_{2k-1}(\lambda_1)\right)\left(\lambda_1 - 4 - \frac{c^2}{\lambda_1 - 16} - \sum_{k=2}^{\infty} A_{2k-1}(\lambda_1)\right) = 0. \tag{19}$$

Therefore, we have the following result.

Theorem 2. *If $|c| < 3$, then*

(a) *The first periodic eigenvalues λ_0 and λ_{-1} are the roots of the equation*

$$\lambda^2 - 4\lambda - 2c^2 - \frac{c^2\lambda}{\lambda - 16} - \sum_{k=2}^{\infty} \lambda A_{2k-1}(\lambda) = 0 \tag{20}$$

in the disk $D_1 := \{\lambda \in \mathbb{C} : |\lambda| \leq 2|c| + 4\}$, and the series $\sum_{k=2}^{\infty} A_{2k-1}(\lambda)$ converges uniformly to an analytic function on the disk D_1. Moreover, (20) *has exactly two roots* (*counting multiplicities*) *in D_1, and these roots coincide with the first two eigenvalues λ_0 and λ_{-1} of L_0.*

(b) *The third periodic eigenvalue λ_{+1} is the root of*

$$\lambda - 4 - \frac{c^2}{\lambda - 16} - \sum_{k=2}^{\infty} A_{2k-1}(\lambda) = 0 \tag{21}$$

in the disk D_1. Moreover, (21) *has exactly one root* (*counting multiplicity*) *in D_1, and this root coincides with the third eigenvalue λ_{+1} of L_0.*

Proof. (a) The proof of (a) was given by Veliev [5] for $|c| < 2$. Besides, he gave the spectral analysis of the operators $L_t(q)$, for $t = 0, 1$, and $L(q)$. In this paper, we have derived the same equation by another method of him; equation (20) follows from (19). Now we prove that the statements remain valid for $|c| < 3$. Let $F(\lambda) := \lambda^2 - 4\lambda - 2c^2 = 0$. Then

$$\begin{aligned} |F(\lambda)| &= |\lambda^2 - 4\lambda - 2c^2| \geq |\lambda|^2 - 4|\lambda| - 2|c|^2 \\ &= (2|c| + 4)^2 - 4(2|c| + 4) - 2|c|^2 = 2|c|^2 + 8|c|, \end{aligned}$$

for all $\lambda \in C_1 := \{\lambda \in \mathbb{C} : |\lambda| = 2|c| + 4\}$. Let $G(\lambda)$ be the left-hand side of (20). Estimating the summands of $|A_{2k-1}(\lambda_1)|$ and $|A'_{2k-1}(\lambda_1)|$, we have

$$|A_{2k-1}(\lambda_1)| < \frac{(\frac{3}{2})^{k-2}|c|^{2k}}{|\lambda_1 - 16|^k |\lambda_1 - 36|^{k-1}}, \qquad |A'_{2k-1}(\lambda_1)| < \frac{3(\frac{3}{2})^{k-2}|c|^{2k}}{|\lambda_1 - 16|^{k+1} |\lambda_1 - 36|^{k-1}}.$$

Using the relations $|\lambda_1| \leq 2|c| + 4$, $|\lambda_1 - 16| \geq 12 - 2|c|$ and $|\lambda_1 - 36| \geq 32 - 2|c|$, and then estimating $\sum_{k=2}^{\infty} |A_{2k-1}(\lambda_1)|$ and $\sum_{k=2}^{\infty} |A'_{2k-1}(\lambda_1)|$, for $|c| < 3$, we obtain

$$\begin{aligned}
\sum_{k=2}^{\infty} |A_{2k-1}(\lambda_1)| &< \frac{|c|^4}{(12-2|c|)^2(32-2|c|)} + \frac{(3/2)|c|^6}{(12-2|c|)^3(32-2|c|)^2} + \frac{(3/2)^2|c|^8}{(12-2|c|)^4(32-2|c|)^3} + \cdots \\
&= \frac{|c|^4}{(12-2|c|)^2(32-2|c|)}\Big(1 + \frac{(3/2)|c|^2}{(12-2|c|)(32-2|c|)} + \frac{(3/2)^2|c|^4}{(12-2|c|)^2(32-2|c|)^2} + \cdots\Big) \\
&= \frac{|c|^4}{(12-2|c|)^2(32-2|c|)} \frac{1}{1 - \frac{3|c|^2}{2(12-2|c|)(32-2|c|)}} = \frac{2|c|^4}{(12-2|c|)[2(12-2|c|)(32-2|c|) - 3|c|^2]} \\
&< \frac{9}{95} \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=2}^{\infty} |A'_{2k-1}(\lambda_1)| &< \frac{3|c|^4}{(12-2|c|)^3(32-2|c|)} + \frac{3(3/2)|c|^6}{(12-2|c|)^4(32-2|c|)^2} + \frac{3(3/2)^2|c|^8}{(12-2|c|)^5(32-2|c|)^3} + \cdots \\
&= \frac{3|c|^4}{(12-2|c|)^3(32-2|c|)}\Big(1 + \frac{(3/2)|c|^2}{(12-2|c|)(32-2|c|)} + \frac{(3/2)^2|c|^4}{(12-2|c|)^2(32-2|c|)^2} + \cdots\Big) \\
&= \frac{6|c|^4}{(12-2|c|)^2[2(12-2|c|)(32-2|c|) - 3|c|^2]} < \frac{9}{190}. \qquad (23)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|G(\lambda) - F(\lambda)| &= \Big|\frac{c^2\lambda}{\lambda - 16} + \sum_{k=2}^{\infty} \lambda A_{2k-1}(\lambda)\Big| \\
&\leq \frac{|c|^2(2|c|+4)}{12-2|c|} + \frac{2(2|c|+4)|c|^4}{(12-2|c|)[2(12-2|c|)(32-2|c|) - 3|c|^2]} < 15 + \frac{90}{95} < 16
\end{aligned}$$

and hence $|G(\lambda) - F(\lambda)| < |F(\lambda)|$ holds for all $\lambda \in C_1$. Since $F(\lambda)$ has two roots in the disk D_1, it follows by Rouche's theorem that $G(\lambda)$ has two roots in D_1. as well These roots coincide with the first two periodic eigenvalues λ_0 and λ_{-1} [5, Theorem 9]. Moreover, by (22) and (23), the series $\sum_{k=2}^{\infty} A_{2k-1}(\lambda)$ converges uniformly to an analytic function on the disk D_1.

(b) The proof of (b) is similar to the proof of (a). Equation (21) follows from (19). Let $g(\lambda)$ be the left-hand side of (21) and $h(\lambda) = \lambda - 4$. Then

$$|h(\lambda)| = |\lambda - 4| \geq |\lambda| - 4 = 2|c| + 4 - 4 = 2|c|,$$

for all $\lambda \in C_1$ and

$$\begin{aligned}
|g(\lambda) - h(\lambda)| &= \left|\frac{c^2}{\lambda - 16} + \sum_{k=2}^{\infty} A_{2k-1}(\lambda)\right| \\
&\leq \frac{|c|^2}{12-2|c|} + \frac{2|c|^4}{(12-2|c|)[2(12-2|c|)(32-2|c|) - 3|c|^2]} < \frac{3}{2} + \frac{9}{95} < \frac{8}{5}.
\end{aligned}$$

Therefore $|g(\lambda) - h(\lambda)| < |h(\lambda)|$ holds for all $\lambda \in C_1$. Since $h(\lambda)$ has one root in the disk D_1, it follows by Rouche's theorem that $g(\lambda)$ has one root in D_1 as well. The other parts of the proof are the same as those for (a). □

Before stating the remark about the nth critical point $V = V_n$, we give the following definitions and remarks from [5], [6], [10].

Definition 1. A number $V = V_2$ is called the second critical point for the first periodic eigenvalue if the first real periodic eigenvalue is a double eigenvalue, that is $\lambda_0 = \lambda_{-1}$ [5, Definition 3].

It was stated and proved [4], [5] that if the number V in potential (3) is less than $\sqrt{5}/2$ and greater than the second critical point V_2, then the first periodic eigenvalues λ_0 and λ_{-1} are nonreal and they are complex conjugate numbers. If $V < V_2$, then λ_0 and λ_{-1} are real and distinct numbers.

Remark 1. It was proved by Veliev [5, Theorem 10] that $0.8884370025 < V_2 < 0.8884370117$. In this case, c^2 changes from -2.15728123 to -2.157281295. We assume that the conjectures about the critical points $V = V_n$, for $n = 3, 4, \dots$, given by Veliev [6] are valid. By the same way, they define the third critical point $V = V_3$ and in general, the nth critical point $V = V_n$, for $n = 2, 3, \dots$ (see also [10, p. 286]). We state [5, Conclusion 1] in our notation: If $V = V_3$, then the third and fourth periodic eigenvalues λ_{+1} and λ_{-2} are equal and real; if $V_2 < V < V_3$, then λ_{+1} and λ_{-2} are real and distinct numbers and if $V > V_3$, then λ_{+1} and λ_{-2} are complex conjugate numbers. In general, if $V = V_n$, then the $(2n-3)$th and $(2n-2)$th periodic eigenvalues λ_{n-2} and λ_{-n+1} are equal and real; if $V_{n-1} < V < V_n$, then λ_{n-2} and λ_{-n+1} are real and distinct numbers; if $V > V_n$, then λ_{n-2} and λ_{-n+1} are complex conjugate numbers and all other periodic eigenvalues after the $(2n-2)$th eigenvalue are real [5, p. 30].

In [5, p. 30], Veliev sketched the shape of the spectrum of the Hill operator with the optical potential (3), for different values of V. In these figures, the end points of the bands of the spectrum are periodic and antiperiodic eigenvalues which are indicated by blue and yellow points, respectively. He stated [5, Conclusion 1, p. 31] that for $V > V_k$, $k = 2, 3, \dots$, no part of the $(2k-3)$rd and $(2k-2)$nd bands, i.e., no part of $\Omega_{k-1} = \Gamma_{2k-3} \cup \Gamma_{2k-2}$, remains real. He also pointed out [5, Conclusion 2, p. 31] that if $V = V_{k+1}$, then the sets $\Omega_1, \Omega_2, \dots, \Omega_{k-1}$ have the shapes as in the third part of Fig. 6 in [5, p. 30]. From these figures and conjectures in [5], we conclude that if $V = V_{k+1}$, then the first $(2k-2)$ periodic eigenvalues are nonreal and pairwise conjugate numbers. Assuming the existence of the critical points from [5], [6], [10], we observe the following:

By the equations obtained in Theorem 1 and Theorem 2, there is no possibility that the $(2n)$th and $(2n+1)$th periodic eigenvalues λ_{-n} and λ_{+n} coincide, for $n = 1, 2, \dots$. However, the first and second eigenvalues coincide for $V = V_2$ and $(2n-1)$st and $(2n)$th periodic eigenvalues may coincide for $n = 2, 3, \dots$ if the radii of the disks defined in Theorem 1 and Theorem 2 increase. However, we cannot calculate these double eigenvalues with our estimations, because the disks D_n and D_{n+1}, for $n = 1, 2, \dots$, do not intersect in our estimations. If the third and fourth periodic eigenvalues are equal, namely if $\lambda_{+1} = \lambda_{-2}$, then, by the inequalities $|\lambda_{+1}| \leq 2|c| + 4$ and $|\lambda_{-2}| \geq 16 - 2|c|$, we have the inequality $2|c| + 4 \geq 16 - 2|c|$, which implies $|c| \geq 3$. Using the relation $c = \sqrt{1 - 4V^2}$, we obtain $V_3 \geq \sqrt{10}/2$. Similarly, if the $(2n+1)$th and $(2n+2)$th periodic eigenvalues are equal, i.e. $\lambda_{+n} = \lambda_{-(n+1)}$, then we have $(2n)^2 + 2|c| \geq (2(n+1))^2 - 2|c|$, since $|\lambda_{+n}| \leq (2n)^2 + 2|c|$ and $|\lambda_{-(n+1)}| \geq (2(n+1))^2 - 2|c|$, for $n \geq s$ and $s = 2, 3, \dots$. In this case, we obtain $|c| \geq (2n+1)$ and $V_{n+2} \geq \sqrt{(2n+1)^2 + 1}/2$, for $n \geq s$ and $s = 2, 3, \dots$. Thus, we stress the following estimation:

Remark 2. The $(n+2)$th critical point $V = V_{n+2}$, defined in [5], [10], as the point at which the $(2n+1)$th and $(2n+2)$th periodic eigenvalues are equal and real, satisfies the inequality

$$V_{n+2} \geq \frac{\sqrt{(2n+1)^2 + 1}}{2},$$

for $n = 1, 2, \dots$, under the assumption that the conjectures about the critical points given by Veliev [6] hold. Obviously, $\lim_{n\to\infty} V_n = \infty$ as stated by Veliev [6, Conjecture 3].

By Theorem 1 and Theorem 2, if $|c| < 3$ which corresponds to the case $V < \sqrt{10}/2$, then all periodic eigenvalues after the second eigenvalue can be calculated numerically as real numbers, with a very small error as in our numerical example. In this case, the first two periodic eigenvalues coincide for a specific value of c^2 between -2.15728123 and -2.157281295, which corresponds to $0.8884370025 < V_2 < 0.8884370117$ and they can be calculated numerically by Theorem 2. If

$V_2 < V < \sqrt{10}/2$, then the first two periodic eigenvalues can be calculated numerically as complex conjugate numbers by Theorem 2.

Similarly, if $3 \le |c| < 5$ or correspondingly $\sqrt{10}/2 \le V < \sqrt{26}/2$, then all periodic eigenvalues after the fifth eigenvalue can be calculated numerically as real numbers by Theorem 1. In this case, the third and fourth eigenvalues coincide and so, the third critical point $V = V_3$ occurs, but we can't calculate this double eigenvalue numerically with our equations (For $|c| = 3$, the fourth and fifth eigenvalues can also be calculated by Theorem 1).

In general, if $(2n-1) \le |c| < (2n+1)$ or correspondingly

$$\frac{\sqrt{(2n-1)^2+1}}{2} \le V < \frac{\sqrt{(2n+1)^2+1}}{2},$$

then all periodic eigenvalues after the $(2n+1)$th periodic eigenvalue can be calculated numerically as real numbers by Theorem 1 (b). In this case, the $(2n-1)$th and $(2n)$th periodic eigenvalues coincide and the $(n+1)$th critical point $V = V_{n+1}$ occurs, but we can't calculate the eigenvalues less than or equal to the $(2n+1)$th periodic eigenvalue numerically with our equations (For $|c| = 2n-1$, the $(2n)$th and $(2n+1)$th eigenvalues can also be calculated by Theorem 1).

Assuming the existence of the critical points from [5], [6], [10], our calculations give an interval for finding the critical point V_{n+1}, and verify the limit of the increasing sequence $\{V_n : n = 2, 3, \dots\}$ of the critical points approaching infinity.

Now, to estimate eigenvalues numerically, we take finite sums instead of the infinite series in Eqs. (14), (15), (20), and (21). If we consider the mth approximation

$$\lambda^2 - 4\lambda - 2c^2 - \frac{c^2\lambda}{\lambda - 16} - \sum_{k=2}^{m} \lambda A_{2k-1}(\lambda) = 0 \tag{24}$$

for the first periodic eigenvalues λ_0 and λ_{-1}, the mth approximation

$$\lambda - 4 - \frac{c^2}{\lambda - 16} - \sum_{k=2}^{m} A_{2k-1}(\lambda) = 0 \tag{25}$$

for the third periodic eigenvalue λ_{+1}, and the mth approximation

$$\lambda - (2n)^2 - \sum_{k=1}^{m} A_{2k-1}(\lambda) \pm \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=2}^{m} B_{2k-1}(\lambda) = 0 \tag{26}$$

for the other eigenvalues λ_{-n} and λ_{+n} of L_0, then we have the following estimates for the remaining terms:

$$\begin{aligned} \left|\sum_{k=m+1}^{\infty} A_{2k-1}(\lambda_1)\right| &\le \sum_{k=m+1}^{\infty} |A_{2k-1}(\lambda_1)| \\ &< \frac{4}{3}\frac{3^m|c|^{2m+2}}{2^m(12-2|c|)^m(32-2|c|)^{m-1}[2(12-2|c|)(32-2|c|)-3|c|^2]} < 312\left(\frac{9}{104}\right)^m \end{aligned}$$

and

$$\begin{aligned} \left|\sum_{k=m+1}^{\infty} A_{2k-1}(\lambda_n) \pm \left(\frac{q_{-1}}{q_1}\right)^n \sum_{k=m+1}^{\infty} B_{2k-1}(\lambda_n)\right| &\le \sum_{k=m+1}^{\infty} |A_{2k-1}(\lambda_n)| + \left|\frac{q_{-1}}{q_1}\right|^n \sum_{k=m+1}^{\infty} |B_{2k-1}(\lambda_n)| \\ &\le \sum_{k=m+1}^{\infty} |A_{2k-1}(\lambda_2)| + \left|\frac{q_{-1}}{q_1}\right|^2 \sum_{k=m+1}^{\infty} |B_{2k-1}(\lambda_2)| \\ &< \frac{3.2^{m-1}|c|^{2m+2}}{(12-2|c|)^m(16-2|c|)^{m-1}[(12-2|c|)(16-2|c|)-2|c|^2]} < \frac{45}{14}\left(\frac{3}{10}\right)^m, \end{aligned}$$

for $|c| < 3$ and $n \ge 1$. Obviously, we will have better approximations as m grows. Besides, for a fixed m, this method gives better approximations as n grows. Now we approach the eigenvalues by the roots of the polynomials derived from the mth approximations (24), (25), and (26), as it was done in [5]. For example, for $n=1$ and $m=3$, we have the third approximations

$$Q_1(\lambda) := \lambda^2 - 4\lambda - 2c^2 - \frac{c^2\lambda}{\lambda-16} - \frac{c^4\lambda}{(\lambda-16)^2(\lambda-36)} - \frac{c^6\lambda}{(\lambda-16)^2(\lambda-36)^2(\lambda-64)} - \frac{c^6\lambda}{(\lambda-16)^3(\lambda-36)^2} = 0,$$

and

$$Q_{-1}(\lambda) := \lambda - 4 - \frac{c^2}{\lambda-16} - \frac{c^4}{(\lambda-16)^2(\lambda-36)} - \frac{c^6}{(\lambda-16)^2(\lambda-36)^2(\lambda-64)} - \frac{c^6}{(\lambda-16)^3(\lambda-36)^2} = 0. \tag{27}$$

Then

$$P_1(\lambda) := (\lambda-16)^3(\lambda-36)^2(\lambda-64)Q_1(\lambda)$$

and

$$P_{-1}(\lambda) := (\lambda-16)^3(\lambda-36)^2(\lambda-64)Q_{-1}(\lambda) \tag{28}$$

are polynomials of degree 8 and 7, respectively. By the same token, we can derive polynomials to approximate the periodic eigenvalues, for $n \geq 2$.

Now consider the operator $L_1(q)$ associated with the antiperiodic boundary conditions. Using the similar formulas

$$(\mu_N - (2n-1)^2)(\Phi_N, e^{i(2n-1)x}) = (q\Phi_N, e^{i(2n-1)x}), \tag{29}$$

$$(\mu_N - (2n-1)^2)(\Phi_N, e^{-i(2n-1)x}) = (q\Phi_N, e^{-i(2n-1)x}), \tag{30}$$

to (9), (10), and then iterating equation (29) m times, we obtain

$$\left(\mu_n - (2n-1)^2 - \sum_{k=1}^{m} a_k(\mu_n)\right)(\Phi_N, e^{i(2n-1)x}) - \left(q_{2n-1} + \sum_{k=1}^{m} b_k(\mu_n)\right)(\Phi_N, e^{-i(2n-1)x}) = r_m(\mu_n),$$

where

$$a_k(\mu_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\dots-n_k}}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\dots-n_k)-1)^2]},$$

$$b_k(\mu_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{2n-1-n_1-n_2-\dots-n_k}}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\dots-n_k)-1)^2]},$$

$$r_m(\mu_n) = \sum_{n_1,n_2,\dots,n_{m+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}(q\Phi_N, e^{i(2(n-n_1-\dots-n_{m+1})-1)x})}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\dots-n_{m+1})-1)^2]}.$$

Here, the sums are taken under the conditions $n_s = \pm 1$, $\sum_{j=1}^{s} n_j \neq 0, 2n-1$ for $s = 1, 2, \dots, m+1$.

Similarly, iterating equation (30) m times, we obtain

$$\left(\mu_n - (2n-1)^2 - \sum_{k=1}^{m} a_k^*(\mu_n)\right)(\Phi_N, e^{-i(2n-1)x}) - \left(q_{-2n+1} + \sum_{k=1}^{m} b_k^*(\mu_n)\right)(\Phi_N, e^{i(2n-1)x}) = r_m^*(\mu_n),$$

where

$$a_k^*(\mu_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\dots-n_k}}{[\mu_n-(2(n+n_1)-1)^2]\cdots[\mu_n-(2(n+n_1+\dots+n_k)-1)^2]},$$

$$b_k^*(\mu_n) = \sum_{n_1,n_2,\dots,n_k} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-2n+1-n_1-n_2-\dots-n_k}}{[\mu_n-(2(n+n_1)-1)^2]\cdots[\mu_n-(2(n+n_1+\dots+n_k)-1)^2]},$$

$$r_m^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_m} q_{n_{m+1}} (q\Phi_N, e^{-i(2(n+n_1+\cdots+n_{m+1})-1)x})}{[\mu_n - (2(n+n_1)-1)^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_{m+1})-1)^2]}.$$

Here, the sums are taken under the conditions $n_s = \pm 1$, $\sum_{j=1}^{s} n_j \neq 0, -2n+1$ for $s = 1, 2, \ldots, m+1$. Therefore, the analogous formulas to (16), (17) and (18) are

$$\left(\mu_n - (2n-1)^2 - \sum_{k=1}^{\infty} a_k(\mu_n)\right)(\Phi_N, e^{i(2n-1)x}) = \left(q_{2n-1} + \sum_{k=1}^{\infty} b_k(\mu_n)\right)(\Phi_N, e^{-i(2n-1)x}),$$

$$\left(\mu_n - (2n-1)^2 - \sum_{k=1}^{\infty} a_k^*(\mu_n)\right)(\Phi_N, e^{-i(2n-1)x}) = \left(q_{-2n+1} + \sum_{k=1}^{\infty} b_k^*(\mu_n)\right)(\Phi_N, e^{i(2n-1)x})$$

and

$$\left(\mu_n - (2n-1)^2 - \sum_{k=1}^{\infty} a_k(\mu_n)\right)^2 = \left(\frac{q_{-1}}{q_1}\right)^{2n-1} \left(q_{2n-1} + \sum_{k=1}^{\infty} b_k(\mu_n)\right)^2, \qquad n \geq 1,$$

respectively. Now we state theorems similar to Theorem 1 and Theorem 2 for the operator $L_1(q)$.

Theorem 3. (a) *If $|c| < 3$ and $n \geq 3$, then $\mu_{\pm n}$ is an eigenvalue of L_1 if and only if it is either the root of the equation*

$$\mu - (2n-1)^2 - \sum_{k=1}^{\infty} a_{2k-1}(\mu) - \left(\frac{q_{-1}}{q_1}\right)^{n-1/2} \sum_{k=2}^{\infty} b_{2k}(\mu) = 0 \tag{31}$$

or the root of

$$\mu - (2n-1)^2 - \sum_{k=1}^{\infty} a_{2k-1}(\mu) + \left(\frac{q_{-1}}{q_1}\right)^{n-1/2} \sum_{k=2}^{\infty} b_{2k}(\mu) = 0 \tag{32}$$

in the disk $d_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| \leq 2|c|\}$, and each of the series in these equations converges uniformly to an analytic function on the disk d_n. Moreover, the roots of (31) *and* (32) *in d_n coincide with the eigenvalues μ_{-n} and μ_{+n} of L_1.*

(b) *In the case of $n = 2$, the statements in* (a) *remain valid for $|c| < 2$.*

Here, we note that in the cases $n = 1$ and $n = 2$, a lemma similar to Lemma 1 will be valid for $|c| < 2$, because $|\mu_1 - 9| \geq 9 - |\mu_1| \geq 8 - 2|c|$ and $|\mu_2 - 1| \geq |\mu_2| - 1 \geq 8 - 2|c|$ by (8). Now for $n = 1$, we have the following theorem.

Theorem 4. *If $|c| < 2$, then $\mu_{\pm 1}$ is an eigenvalue of L_1 if and only if it is either the root of the equation*

$$\mu - 1 - c - \sum_{k=1}^{\infty} a_{2k-1}(\mu) = 0 \tag{33}$$

or the root of

$$\mu - 1 + c - \sum_{k=1}^{\infty} a_{2k-1}(\mu) = 0 \tag{34}$$

in the disk $d_1 := \{\mu \in \mathbb{C} : |\mu| \leq 2|c| + 1\}$, and each of the series in these equations converges uniformly to an analytic function on the disk d_1. Moreover, the roots of (33) *and* (34) *in d_1 coincide with the first antiperiodic eigenvalues μ_{-1} and μ_{+1}.*

Now let us approach the antiperiodic eigenvalues by the polynomials derived from the mth approximations of (31)-(34). For $n = 1$, $m = 3$, and $j = 1, 2$, we have

$$H_j(\mu) := \mu - 1 + (-1)^j c - \frac{c^2}{\mu - 9} - \frac{c^4}{(\mu-9)^2(\mu-25)} - \frac{c^6}{(\mu-9)^2(\mu-25)^2(\mu-49)} - \frac{c^6}{(\mu-9)^3(\mu-25)^2} = 0. \tag{35}$$

Then

$$S_j(\mu) := (\mu-9)^3(\mu-25)^2(\mu-49)H_j(\mu) \tag{36}$$

is a polynomial of degree 7. By the same token, we can derive polynomials to approximate the antiperiodic eigenvalues, for $n \geq 2$. Now we present a numerical example.

Example 1. For $m = 3$ and $c^2 = -2.157281295$, Veliev [5] approximated the first periodic eigenvalues λ_0 and λ_{-1}. Now we have the following approximations to the third periodic eigenvalue λ_{+1} and the first antiperiodic eigenvalues μ_{-1} and μ_{+1}:

First, we show that λ_{+1} is the real eigenvalue lying inside the circle

$$C = \{\lambda \in \mathbb{C} : |\lambda - 4.1814942277| = 1.7 \times 10^{-6}\}.$$

The root of the polynomial $P_{-1}(\lambda)$ defined by (28), lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|c| + 4\}$, is $r_1 = 4.1814942277$. The other roots of $P_{-1}(\lambda)$ are

$r_2 = 15.8535021182$, $r_3 = (15.9823184944 - 0.119095369803i)$,
$r_4 = (15.9823184944 + 0.119095369803i)$, $r_5 = (36.000183379 - 0.00333664975667i)$,
$r_6 = (36.000183379 + 0.00333664975667i)$, $r_7 = 63.9999999074$.

Using the decomposition

$$Q_{-1}(\lambda) = \frac{(\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_7)}{(\lambda-16)^3(\lambda-36)^2(\lambda-64)},$$

we obtain by direct calculation $|Q_{-1}(\lambda)| > 1.8496 \times 10^{-7}$, for all $\lambda \in C$. On the other hand, again by straightforward calculations, we have $\sum_{k=4}^{\infty} |A_{2k-1}(\lambda)| < 1.8269 \times 10^{-7}$, for all $\lambda \in C$. Therefore, by Rouche's theorem, equation (21) has only one root inside the circle C. Thus, using Theorem 2 (b) and the spectral analysis of L_0 given by Veliev [5], we conclude that λ_{+1} is the real eigenvalue lying inside the circle C.

Now we show that μ_{-1} and μ_{+1} are the complex eigenvalues lying inside the circles

$$\delta_1 = \{\mu \in \mathbb{C} : |\mu - (1.26575008922 - 1.52020432568i)| = 1.4 \times 10^{-5}\}$$

and

$$\delta_2 = \{\mu \in \mathbb{C} : |\mu - (1.26575008922 + 1.52020432568i)| = 1.4 \times 10^{-5}\},$$

respectively. The roots (36) of the polynomials $S_1(\mu)$ and $S_2(\mu)$ in the disk $d_1 = \{\mu \in \mathbb{C} : |\mu| \leq 2|c| + 1\}$ are

$x_1 = (1.26575008922 + 1.52020432568i)$, $y_1 = (1.26575008922 - 1.52020432568i)$,

respectively. The other roots of $S_1(\mu)$ are

$x_2 = (8.96777697119 + 0.142338162679i)$, $x_3 = (8.79563202223 - 0.0317230792875i)$,
$x_4 = (8.97007606112 - 0.162097407292i)$, $x_5 = (25.0005579806 - 0.00577397577187i)$,
$x_6 = (25.0002071021 + 0.00582061314113i)$ $x_7 = (48.9999997735 - 0.00000000692262634543i)$,

and the other roots of $S_2(\mu)$ are

$y_2 = (8.96777697119 - 0.142338162679i)$, $y_3 = (8.79563202223 + 0.0317230792875i)$,

$y_4 = (8.97007606112 + 0.162097407292i)$, $y_5 = (25.0005579806 + 0.00577397577187i)$,
$y_6 = (25.0002071021 - 0.00582061314113i)$ $y_7 = (48.9999997735 + 0.0000000692262634543i)$.

Using the decompositions

$$H_1(\mu) = \frac{(\mu - x_1)(\mu - x_2)\cdots(\mu - x_7)}{(\mu - 9)^3(\mu - 25)^2(\mu - 49)}$$

and

$$H_2(\mu) = \frac{(\mu - y_1)(\mu - y_2)\cdots(\mu - y_7)}{(\mu - 9)^3(\mu - 25)^2(\mu - 49)},$$

by direct calculations, we obtain $|H_1(\mu)| > 4.6113 \times 10^{-6}$, for all $\mu \in \delta_2$ and $|H_2(\mu)| > 4.6113 \times 10^{-6}$, for all $\mu \in \delta_1$. On the other hand, one can easily calculate that

$$\sum_{k=4}^{\infty} |a_{2k-1}(\mu)| < 4.4786 \times 10^{-6}$$

for all $\mu \in \delta_1 \cup \delta_2$. The proof follows from Rouche's theorem and Theorem 4; each of the equations (33) and (34) has only one root inside the circle δ_2 and δ_1, respectively and μ_{-1} and μ_{+1} are the complex eigenvalues lying inside δ_1 and δ_2, respectively.

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CONFLICT OF INTEREST

The author of this work declares that she has no conflicts of interest.

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